# Extended mean-field study of a stochastic cellular automaton

György Szabó
*Research Institute for Materials Science, POB 49, H-1525 Budapest, Hungary*

Géza Ódor*
*Department of Physics, University of Illinois at Chicago, Chicago, Illinois 60607-7059*
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Extended mean-field theory is used for studying the one-dimensional stochastic cellular automaton with Rule 18 defined by Wolfram [Rev. Mod. Phys. **55**, 601 (1983)]. The analysis is carried out at different levels taking $n$-point and $n$-pair correlations explicitly into consideration. The pair approximations reproduce the exact results in the deterministic limit. The critical behavior is studied by the Padé approximant method and the predicted critical probability and exponent agree with previous data within a few percent.

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## I. INTRODUCTION

Cellular automata (CA) have general applications in mathematics, nonequilibrium physics, chemistry, and biology [1]. In these systems the dynamics is defined in discrete time steps with rules depending on local neighborhood. Most of the analyses are restricted to stationary states if the system has been started from a random configuration. The local rules may be stochastic. These stochastic CA, even the one-dimensional ones, exhibit continuous phase transitions with universal critical exponents and scaling laws [2]. For example, in a two-state model the site variable may be 0 or 1. Here, the concentration of 1s (henceforth called particles) may decrease continuously when decreasing the probability parameter and the transition to the empty state may be critical. It is known that a large variety of nonequilibrium models including CA belong to the same universality class [2,3]. Recently, however, several authors [4,5] have introduced stochastic CA by permitting random jumps between two steps. In these systems the long range correlations are "washed out" therefore this type of mixing drives the system towards a mean-field behavior which differs significantly from previous ones. Here the application of mean-field theory suggests itself. In a previous paper [6] this method describes how the continuous transition becomes first order when increasing the effect of mixing. Very recently Dickman has discussed the models exhibiting unusual phase diagrams and new kinds of critical behavior [7].

In this work we concentrate on the mean-field analysis of the one-dimensional stochastic cellular automaton with Rule 18 [1]. For this purpose the traditional mean-field analysis [1,8] has been extended to take the 3- and 4-point correlations explicitly into consideration. This technique has already been used successfully to study the stationary states in driven lattice-gas models [9–12].

Grassberger has observed that several deterministic cellular automata exhibit a kind of spontaneous symmetry breaking [13] on an infinite lattice. Namely, one can observe increasing domains in which the particles reside at odd (or even) sites. This behavior is not affected if the creation of a particle (permitted by rule) is occasional. Consequently, the stationary state may be characterized by (00) and (10) [or (00) and (01)] pairs in the thermodynamic limit. This observation has inspired us to develop a mean-field approximation allowing correlations between the neighboring pairs.

The above pair approximations reproduce the exact result [1] in the deterministic limit. The results become more and more accurate if we increase the number of subsequent neighbors taking all the correlations between them into consideration. The situation is analogous to the low temperature series expansions. This is the reason why we have determined the Padé approximants for the investigation of critical behavior. The suggested critical probability and exponent agree very well with previous results and confirm that the model belongs to the universality class of directed percolation (DP) or Reggeon field theory [14].

Section II describes the model and in Sec. III we detail the $n$-point and $n$-pair mean-field analyses. The results of Padé approximation are given in Sec. IV. The last section presents the conclusions.

## II. THE MODEL

We consider a one-dimensional lattice on which the site variables $s_i(t)$ may be 0 or 1 (empty or occupied site) at a discrete time $t$. In the subsequent time $s_i(t+1)$ will be 1 with a probability $p$ if $s_{i-1}(t) = s_i(t) = 0$ and $s_{i+1}(t) = 1$ or $s_{i-1}(t) = 1$ and $s_i(t) = s_{i+1}(t) = 0$, otherwise $s_i(t+1) = 0$. In the limit $p \to 1$ the model is equivalent to the deterministic cellular automaton Rule 18 introduced by Wolfram [1]. In the stationary state

*Permanent address: Research Institute for Materials Science, POB 49, H-1525 Budapest, Hungary.

 

of the deterministic model the average concentration of particles $c = 1/4$ if the system is started from a random configuration. The average concentration decreases with $p$ and vanishes at $p_c$. Below $p_c$ only the empty chain ($c = 0$) is stable.

It is easy to see that the above rule excludes all the configurations in which three subsequent sites are occupied simultaneously. Grassberger has shown that the probability of the simultaneous occupation of two neighboring sites vanishes for large times [13]. In some deterministic models, Grassberger has observed the formation of large domains within particles to be separated by odd empty sites. In other words, inside the domains the particles reside at odd (or even) sites. The boundary between two domains is considered as a kink which walks randomly [15–17] and it may annihilate with an antikink leading to the collapse of a domain. Finally, the density of kink goes to zero and the system remains locally in an "ordered" state. The annihilation of a particle does not produce a new domain therefore the above statements remain valid for stochastic CA too. In the "ordered" state the evolution is equivalent to Rule 90 [1]. Domany and Kinzel [18] have shown that this stochastic CA is equivalent to directed percolation and to an Ising model, i.e., these models belong to the same nonequilibrium universality class. Consequently, the concentration decreases as $c \sim (p - p_c)^\beta$ if the critical probability $p_c$ is approached. The critical exponent $\beta$ is expected to be equivalent to $\beta^{\rm DP} = 0.277 \pm 0.001$ obtained for directed percolation [19]. Results from computer simulations have shown that $\beta^{\rm CA} = 0.285 \pm 0.005$ [4].

## III. MEAN-FIELD THEORY

The particle positions are described by a set of site variables as given above. The present analysis is restricted to the stationary states therefore the notation of time dependence is omitted. In the stationary state the particle distribution is assumed to be symmetric with respect to translation and reflection. In mean-field theory the stationary state is characterized by the probabilities of $n$-point configurations $P_n(s_1, ..., s_n)$ on subsequent sites. These quantities satisfy the following conditions:

$$\begin{aligned} P_n(s_1, ..., s_n) &= \sum_{s_{n+1}} P_{n+1}(s_1, ..., s_n, s_{n+1}) \ , \\ P_n(s_1, ..., s_n) &= \sum_{s_0} P_{n+1}(s_0, s_1, ..., s_n) \ , \\ P_n(s_1, ..., s_n) &= P_n(s_n, ..., s_1) \ . \end{aligned} \tag{1}$$

In principle, $2^n - 1$ parameters are required to define the probability of all the $n$-point configurations. This number, however, is drastically reduced by the above conditions. For example, at a single lattice point

$$P_1(0) = 1 - c \ , \quad P_1(1) = c \ , \tag{2}$$

where $c$ is the average particle concentration. The probabilities of two-point configurations may be given by introducing a single parameter $z$, namely

$$\begin{aligned} P_2(0,0) &= (1-c)^2 + z \ , \\ P_2(1,0) = P_2(0,1) &= c(1-c) - z \ , \\ P_2(1,1) &= c^2 + z \ . \end{aligned} \tag{3}$$

For higher levels we follow the parametrization introduced in a previous paper [11] (further details therein).

The above parameters are determined by a set of nonlinear equations relating block probabilities of the stationary state by the CA rule. According to the above rules, for a one-point configuration

$$P_1(1) = p[P_3(1,0,0) + P_3(0,0,1)] \ , \tag{4}$$

for a two-point configuration ($n = 2$)

$$P_2(1,1) = p^2 P_4(1,0,0,1) \ , \tag{5}$$

and the continuation is straightforward. These equations become more and more complicated with increasing $n$.

At the level of $k$-point approximation the correlations are neglected for $n > k$, that is, $P_n(s_1, ..., s_n)$ is expressed by using the Bayesian extension process [8,11],

$$P_n(s_1, ..., s_n) = \frac{\prod_{j=0}^{j=n-k} P_k(s_{1+j}, ..., s_{k+j})}{\prod_{j=1}^{j=n-k} P_{k-1}(s_{1+j}, ..., s_{k-1+j})} \ . \tag{6}$$

The one-point approximation assumes $z = 0$ in Eq. (3), and Eq. (4) leads to the well known solutions: $c = 0$ and $c = 1 - 1/\sqrt{2p}$ for $p > 1/2$.

In two-point approximation the particle distribution is described by two parameters $c$ and $z$ whose value is determined by Eqs. (4) and (5), which obey the following forms:

$$c = 2p \frac{[c(1-c) - z][(1-c)^2 + z]}{1 - c} \ , \tag{7}$$

$$c^2 + z = p^2 \frac{[c(1-c) - z]^2[(1-c)^2 + z]}{(1-c)^2} \ . \tag{8}$$

Besides the trivial solution ($c = z = 0$), these equations suggest that

$$c = \frac{2\left(2 - 7p + p^2 + p\sqrt{8 - 4p + p^2}\right)}{4 - 20p + 5p^2} \tag{9}$$

and

$$P_2(1,1) = \frac{pc^2}{2(1-c) + pc} \ . \tag{10}$$

It is emphasized that the critical probability is not changed, i.e., $p_c = 1/2$.

For three- and four-point approximations we have four and seven parameters to be determined by solving the corresponding equations. In both cases the critical probability is equal to 2/3. The results of numerical solution are plotted in Fig. 1 in comparison with Monte Carlo data indicated by bullets. This figure demonstrates clearly that the convergence towards the exact result is slow when increasing the level of approximation. The coincidences of $p_c$ at different levels refer to the fact that

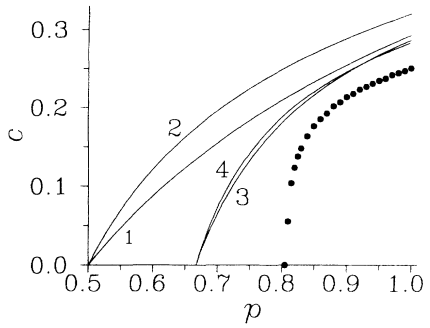


FIG. 1. Average concentration vs $p$ suggested by $n$-point approximations (labels refer to $n$).

the parameters are not equivalent from the viewpoint of criticality.

At the same time $P_2(1,1)$ decreases in agreement with the expectation mentioned above. It is also observed that the probability of those configurations containing neighboring particles or even empty sites between two particles remains finite although these quantities vanish in simulations for long times [13].

The long-time limit, however, may be explicitly taken into consideration. In the stationary state the particles reside at odd sites for odd [even] times and at even sites for even [odd] times. Consequently, the configuration may be composed of (00) and (10) [(00) and (01)] pairs. In $n$-pair approximations our analysis will be restricted to this subset of the configurations. For this purpose we introduce a pair variable $\tilde{s}_i = (s_{2i}, s_{2i+1})$ which may be $\tilde{0} = (0,0)$ or $\tilde{1} = (1,0)$ where the tilde above figures and variables refers to pairs henceforth. It is assumed that the distribution of the mentioned pairs is independent of the parity of time. On the analogy of $P_n(s_1,...,s_n)$ we introduce $\tilde{P}_n(\tilde{s}_1,...,\tilde{s}_n)$ which describes the probability of the given configurations on subsequent sites. For example, the two pair probability $\tilde{P}_2(\tilde{0},\tilde{1})$ is equivalent to $P_4(0,0,1,0)$. The quantities $\tilde{P}_n(\tilde{s}_1,...,\tilde{s}_n)$ also satisfy the conditions (1). Consequently, we can use the same method for determining pair correlations as for point correlations described above. The only difference in parametrization we suggest for practical reasons is the following:

$$\tilde{P}_1(\tilde{1}) = 2c\ ,\quad \tilde{P}_1(\tilde{0}) = 1-2c, \tag{11}$$

where $c$ is the concentration introduced above.

In this subset of configurations the original Rule 18 becomes simpler [1,15,16]. This simplified rule gives modified relations among pair configurations. Namely, the probability of the (1,0) pair is related to two-pair probabilities as

$$\tilde{P}_1(\tilde{1}) = p[\tilde{P}_2(\tilde{1},\tilde{0}) + \tilde{P}_2(\tilde{0},\tilde{1})]\ , \tag{12}$$

and

$$\tilde{P}_2(\tilde{1},\tilde{1}) = p^2[\tilde{P}_3(\tilde{1},\tilde{0},\tilde{1}) + \tilde{P}_3(\tilde{0},\tilde{1},\tilde{0})]\ . \tag{13}$$

Further relations may easily be derived for higher levels too. For example, at the level of three-pair configuration we need the following relations:

$$\tilde{P}_3(\tilde{1},\tilde{1},\tilde{1}) = p^3[\tilde{P}_4(\tilde{1},\tilde{0},\tilde{1},\tilde{0}) + \tilde{P}_4(\tilde{0},\tilde{1},\tilde{0},\tilde{1})] \tag{14}$$

and

$$\begin{aligned}\tilde{P}_3(\tilde{1},\tilde{0},\tilde{1}) = {} & p^3(1-p)[\tilde{P}_4(\tilde{1},\tilde{0},\tilde{1},\tilde{0}) + \tilde{P}_4(\tilde{0},\tilde{1},\tilde{0},\tilde{1})] \\ & + p^2[\tilde{P}_4(\tilde{1},\tilde{0},\tilde{0},\tilde{1}) + \tilde{P}_4(\tilde{0},\tilde{1},\tilde{1},\tilde{0})]\ .\end{aligned} \tag{15}$$

The relations are more complicated at higher levels. Notice that the right hand side of these relations does not change when substituting $1-\tilde{s}_i$ for each $\tilde{s}_i$. This symmetry is a striking consequence of the simplified rule mentioned above.

In one-pair approximation we neglect all the correlations appearing between the pairs. In this case the nontrivial solution of Eq. (12) is the following:

$$c = \frac{1}{4}\left(2 - \frac{1}{p}\right)\ . \tag{16}$$

This solution reproduces the exact result in the deterministic limit [$c(p=1) = 1/4$]. In this approximation the critical probability ($p_c = 1/2$) is equivalent to those suggested above by the one- and two-point approximations.

The two-pair approximation takes the correlation ($z$) between the neighboring pairs into consideration. In this case we have two parameters ($c$ and $z$) to be determined by solving Eqs. (12) and (13). The nontrivial solution suggests that the concentration is

$$c = \frac{2-3p}{4-8p} \tag{17}$$

and the first neighbor pair-pair correlation defined by (3) is given as

$$z = \frac{(1-p)^2(3p-2)}{4p(2p-1)}\ . \tag{18}$$

The concentration in the deterministic limit ($p=1$) is the same as for the one-pair approximation case [see Eq. (16)]. This is not surprising because the first neighbor pair correlation vanishes for $p=1$. According to this approximation $p_c = 2/3$ which agrees with the prediction of three- and four-point approximations.

At the level of three-pair approximation the probability of the pair configurations is expressed by four parameters whose value is evaluated by the numerical solution of Eqs. (12)–(15). At the next level we have determined seven parameters using the same method. The suggested critical probabilities are $p_c = 0.7094$ and 0.7413 at these levels. At higher levels, however, the application of this method is limited by the computer capacity. To avoid this difficulty we have concentrated on the effect of relevant parameters. Systematic analysis of the above results shows the existence of irrelevant parameters whose effect on $p_c$ is extremely weak. The existence of such parameters has already been illustrated in Fig. 1. It is found that the relevant parameters (correlations) are related

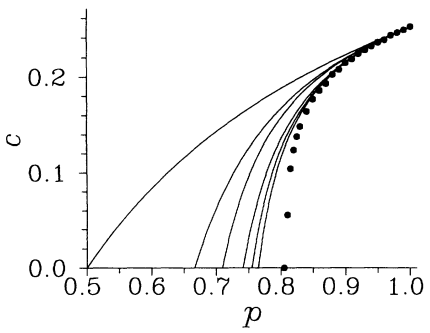


FIG. 2. Average concentration vs $p$ for the level of 1-, 2-, 3-, 4-, 5′-, and 6′-pair approximations (from left to right). The bullets come from simulations.

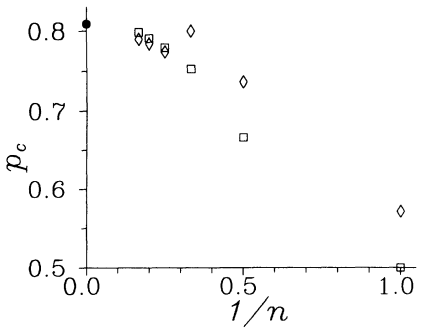


FIG. 3. Dependence upon $1/n$ of critical probability $p_c$ suggested by the Padé approximants $Q_1^1$ (◇) and $Q_2^2$ (□). The bullet represents the simulation result.

to the appearance of large empty regions which become dominant in the limit $c \to 0$. According to Eqs. (1) the configurations $[\tilde{P}_n(\tilde{0},\tilde{0},\ldots,\tilde{0},\tilde{0})]$ are connected linearly with $\tilde{P}_n(\tilde{1},\tilde{0},\ldots,\tilde{0},\tilde{1})$ and $\tilde{P}_n(\tilde{1},\tilde{0},\ldots,\tilde{0},\tilde{0})$, where dots indicate empty regions [11]. These quantities exhibit power law behavior in the close vicinity of $p_c$. Introducing one (two) additional parameter(s) we could extend our analysis to the level of 5′- and 6′-pair approximations where the prime refers to reduction in the number of parameters. More precisely, we have taken all the parameters appearing in $\tilde{P}_5(\tilde{0},\tilde{0},\tilde{0},\tilde{0},\tilde{0})$ and $\tilde{P}_6(\tilde{0},\tilde{0},\tilde{0},\tilde{0},\tilde{0},\tilde{0})$ into consideration while the remaining correlations of 5- and 6-pair approximations are neglected. At the level of 5′- and 6′-pair approximations we have determined the $p$ dependence of eight and nine parameters, and the numerical calculations suggest $p_c = 0.7551$ and 0.7650, respectively.

The results of pair approximations are summarized in Fig. 2. As expected, the convergence of pair approximations towards the exact result is faster in comparison with $n$-point approximations. For all the levels $c \to p/4$ if $p \to 1$. The Taylor series of $c$ with respect to $(1-p)$ may be written as

$$c = \tfrac{1}{4}\sum_{k=0}^{\infty} a_k (1-p)^k, \tag{19}$$

where $a_0 = -a_1 = 1$ and the values of coefficients $a_k$ $(k > 1)$ depend on the level of pair approximation. Equations (16) and (17) determine the values of $a_k$ at the level of one- and two-pair approximation. For higher levels, however, the accuracy of numerical calculation limits the number of $a_k$ available. For example, $a_2 = -3$, $a_3 = -7$, $a_4 = -21$, and $a_5 = -67$ in the case of four-pair approximation. In the knowledge of these coefficients we are allowed to determine the Padé approximants.

## IV. PADÉ APPROXIMANTS

The method of Padé approximants proved to be a successful tool in obtaining the critical temperature and exponents for the Ising model. This method continues analytically the truncated power series beyond the radius of convergence (for detailed description see [20]). The Padé approximant $Q_D^N(x)$ to the function $F(x)$ is simply the ratio of two polynomials of orders $N$ and $D$,

$$Q_D^N(x) = \frac{n_0 + n_1 x + \cdots + n_N x^N}{1 + d_1 x + \cdots + d_D x^D}. \tag{20}$$

Here we choose the function $F(x)$ to be the logarithmic derivative series of Eq. (19) because then the power law singularity of $c$ obeys the following form:

$$F(x) = \sum_{k=0}^{\infty} b_k x^k \sim \frac{\beta}{x - x_c}, \tag{21}$$

where $x = 1 - p$. This logarithmic derivative series has a simple pole at $x_c = 1 - p_c$ with a residue equal to $\beta$. In other words, if we determine the Padé approximants to the logarithmic derivative series of (19) then one of the roots of the denominator polynomial gives a prediction for $p_c$ and the corresponding residue is the critical exponent $\beta$.

Following the work by Baker [20] we have determined the Padé approximants $Q_1^1$ and $Q_2^2$ for each level of the $n$-pair approximation. The evaluation of $Q_2^2$ demands the determination of five coefficients in Eq. (19). Due to the numerical errors the additional coefficients are accessible with low accuracy precluding the determination of the Padé approximants for higher orders.

The suggested critical probabilities and $\beta$ exponents are plotted versus $1/n$ in Figs. 3 and 4. As a comparison

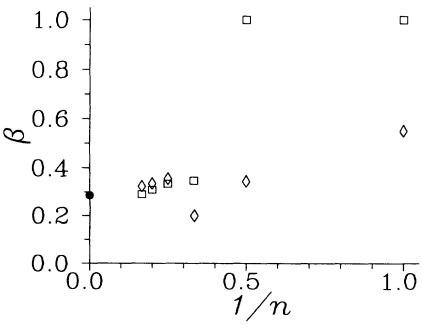


FIG. 4. Critical exponents vs $1/n$ suggested by the Padé approximants $Q_1^1$ (◇) and $Q_2^2$ (□).

the bullets represent the simulation data [4]. Notice that $Q_2^2$ reproduces the logarithmic derivative of (16) and (17) for $n = 1$ and 2. If $n \geq 4$ then the Padé approximants $Q_1^1$ and $Q_2^2$ result in similar data. For example, $Q_2^2$ suggests $p_c = 0.7795$ and $\beta = 0.3355$ at the level of 4-pair approximation. As expected, the best results are obtained for the level of 6′-pair approximation: $p_c = 0.7986$ and $\beta = 0.2900$.

## V. CONCLUSIONS

Systematic mean-field analysis was performed to study the stochastic cellular automaton with Rule 18. The traditional mean-field analysis is extended to take into consideration the effect of $n$-point and $n$-pair correlations on subsequent sites. In $n$-point approximation the solution tends slowly towards the exact result when increasing $n$.

The long-time correlations found by Grassberger reduce the number of possible configurations drastically. The $n$-pair approximations are based on the finding that the stationary state may be built from (0,0) and (1,0) pairs. This simplification summarizes a portion of contributions coming from $n$-point correlations for arbitrary large $n$. Consequently, the $n$-pair approximations predict more accurate results in comparison to $n$-point approximations. In fact, the pair approximations reproduce the exact result in the deterministic limit ($p \to 1$). In this limit the correlations disappear between the pairs in agreement with previous results.

As a means of studying the critical behavior, our mean-field analysis is extended by determining the Padé approximants to the results of $n$-pair approximation. This method suggests values for the critical probability and $\beta$ exponent in close agreement with data obtained from simulations. In the light of the present results one can think that the Padé approximants of low order fit the exact solution extremely well.

In summary, the present mean-field analysis is proved to be a useful tool for investigating stationary states in a stochastic CA. This method may be easily adapted for other CA including the consideration of the effect of mixing.

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* Permanent address: Research Institute for Materials Science, POB 49, H-1525 Budapest, Hungary.

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