Derivation of potential for group interactions in evolutionary games

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For spatial evolutionary games, a potential can be derived if the interaction is described by two-player, two-strategy, symmetric games that yield thermodynamical behavior for a suitable dynamical rule. Here, we show the existence of a potential for two-strategy group interactions in which the players with identical strategies receive equal payoffs. This type of group interaction includes some extended versions of the public goods game. Some peculiar consequences of these group interactions are illustrated in a simple model in which the players are located at the sites of a square lattice with periodic boundary conditions. This five-player group interaction supports the formation of striplike arrangements of the two strategies. Monte Carlo simulations, however, indicate the existence of many other ground states and the relevance of frustration in shaping the macroscopic behavior when the effect of noise is taken into consideration.

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I. INTRODUCTION

The application of the methods of statistical physics in the investigation of multiagent social [1–3] and biological [4–6] systems was initiated by the successful adaptation of the concepts and methods introduced for the investigation of Ising-type models [7–9]. In the original Ising models, the atoms that are in either a $+1$ or a -1 spin state are arranged in a crystal structure, and their attractive nearest-neighbor interactions result in the formation of ordered spin states at low temperatures that transform into a disordered macroscopic phase if the temperature exceeds a critical value. Similar phenomena can also occur in spatial evolutionary games in which the two-player interactions, the connectivity networks, and the dynamical rules are similar.

In game theory [10], however, group interactions have also been considered for a long time because these play an important role in the macroscopic behavior of social [11–16] and biological [17–20] systems. One of the most well-investigated examples is the public goods game (PGG) in which n players independently decide whether to pay a cost c for a benefit the other players enjoy, too. In this game, the total investment (the sum of the cost paid by the cooperative players) is multiplied by a factor of r ($r < n$) and then shared equally among the players. This game represents a social dilemma, because their individual interests dictate that they choose to defect (not to pay the cost) to all players, who ultimately receive nothing as a result. This group interaction can be built up as a sum of symmetric two-player two-strategy games between all pairs of players [21–23]. Thus, this system has a potential [24–26] and exhibits thermodynamical behavior [24,27] for a suitable dynamical rule and an infinitely large number of players. At the same time, some generalized versions of the public goods game (e.g., threshold public goods games) cannot be considered as a sum of two-player games [28–32]; consequently, the existence of a potential is not guaranteed.

In the next section, we show that a potential can be derived for the generalized PGGs introduced by Broom, *et al.* [22] and also discussed by Li, *et al.* [23]. In these group interactions, the players are not distinguishable. More precisely, players with identical strategies get the same payoff, similarly to many-particle systems. These group interactions, however, can exhibit a wide range of interesting behaviors if the players are distributed on a square lattice (or any other connectivity network) and their total income comes from several group interactions. In Sec. III we consider a simple spatial model that exhibits many optimal strategy arrangements (or, in other words, strict Nash equilibria). Using the logit rule [24,33,34], we study the stationary states via Monte Carlo simulations. As part of these analyses, we found interesting combinations of phenomena related to the existence of many optimal strategy arrangements and the presence of frustration, which have already been observed and studied in minority games [35–38], a simple subset of group interactions. We survey these complex phenomena, whose components were analyzed previously in several physical systems involving four-spin interactions [39,40] or relationships among ordering, metastability, and topological features [41–44].

II. POTENTIAL FOR GROUP INTERACTIONS

Now, we turn to the study of a generalized public goods game discussed by Refs. [22,23] in which n players have to choose between strategies A and B independently of the others. In these models, the players with strategy A or B receive payoffs a_i and b_i ($i = 0, 1, ..., n$) respectively, if strategy B is chosen by i players. Evidently, in this notation, the values of a_n and b_0 are irrelevant parameters. We have to emphasize that this group interaction model is characterized by $2(n-1)$ payoff parameters, while we can distinguish 2^n strategy distributions $\mathbf{s} = (s_1, s_2, ..., s_n)$ defined by $s_k = A$ or B for player k ($k = 1, 2, ..., n$) within the group.

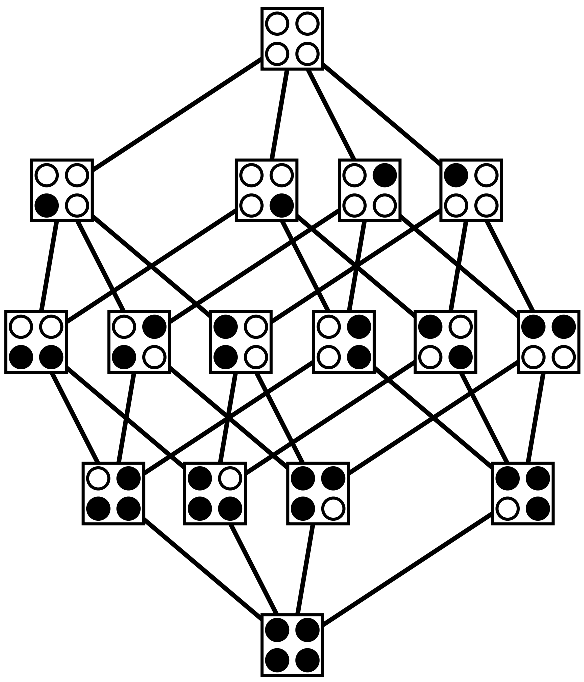

FIG. 1. Dynamical graph for two-strategy group interactions with four players. The possible strategy distributions are represented by black and white circles standing in for individual A and B strategy choices, respectively.

To derive and investigate the potential, we use the concept of dynamical graphs, introduced by Schnakenberg [45], the nodes of which represent the strategy distributions (or microscopic states in statistical physics) and edges connect nodes that differ only in a single player's strategy choice. In other words, the edges denote transitions generated by unilateral strategy changes between the possible strategy distributions.

The corresponding dynamical graph can be visualized by a suitable two-dimensional projection of an n-dimensional hypercube (Fig. 1 shows the $n = 4$ case). In this graph representation, the uppermost node represents the single state where all players choose strategy A ($i = 0$) and receive identical payoff a_0. For $i = 1$ the states are arranged horizontally and connected to the node in the previous $i = 0$ layer by edges identifying the unilateral strategy changes that transfer the corresponding strategy distributions into each other. In the next row of strategy distributions, $i = 2$ and the edges are obtained similarly. This process can be repeated, until all strategy distributions and unilateral strategy changes are enumerated. Note that along the edges between two neighboring rows of strategy distributions, the incentives driving the unilateral strategy changes are identical. Quantitatively, the payoff increase of the active player is $(b_{i+1} - a_i)$ if $i \to (i + 1)$. This feature simplifies the derivation of the potential $V(\mathbf{s})$ that sums up the driving forces of the active players for consecutive unilateral strategy changes [24–27]. Namely, for this group interaction, the values of the potential depend on the strategy distribution $\mathbf{s}$ only through the composition of the group as represented by i, the number of players choosing strategy A, and are given by the function

$$v(i) = \sum_{j=i}^{n-1} a_j + \sum_{j=1}^{i} b_j, \tag{1}$$

for $0 \leqslant i \leqslant n$ when choosing

$$v(0) = \sum_{j=0}^{n-1} a_j. \tag{2}$$

As a result,

$$v(n) = \sum_{j=1}^{n} b_j. \tag{3}$$

It is worth mentioning that the existence of the potential is intimately related to the requirement that the individual driving forces add up to zero along all directed closed loops on the dynamical graph. This condition is trivially satisfied for these games, because the closed loops include forward and backward transition edges between states of two neighboring rows $[i \leftrightarrow (i + 1)]$. Furthermore, the four-edge loops (representing a side of the hypercube) in these dynamical graphs define a symmetric two-strategy game between the two active players, and for this type of interaction the existence of a potential is guaranteed [27].

Using this parametrization, the traditional public goods game is reproduced when strategy A and B describe defective and cooperative behaviors with their payoffs expressed as $a_i = irc/n$ and $b_i = a_i - c$. For this interaction each player with strategy B has an incentive [a payoff increase of $c(1 - r/n)$] to modify her strategy to A. Consequently, if $v(0) = 0$ is chosen, then $v(i) = -ic(1 - r/n)$. Due to this feature, traditional public goods games have a single strict Nash equilibrium in which all selfish players choose strategy A.

Other simple examples are partnership [46] (or fraternal [27]) games, in which the players receive equal payoffs [independently of the chosen strategy, that is, $a_i = b_i$ if $0 < i < n$] for all strategy distributions. Interestingly, for these interactions the potential can be given as $v(i) = a_i$ (if $0 \leqslant i < n$) and $v(n) = b_n$, which directly relates the potential to the total income of the players $[nv(i)]$.

In general, the highest value of the potential $[\max V(\mathbf{s})]$ determines an optimal strict Nash equilibrium. For these partnership games, however, the system can possess many Nash equilibria if $\max V(\mathbf{s})$ is reached for some $0 < i < n$. Even more interesting examples are those types of interactions for which the $v(i)$ function itself has two (or more) local maxima. Further peculiar consequences of these interactions occur for spatial evolutionary games we discuss in the next section.

III. A SPATIAL MODEL

Now, we consider a spatial evolutionary game in which the players are located at the sites of a square lattice with $L \times L$ sites. We assume periodic boundary conditions to preserve translation invariance. In this spatial evolutionary game, the payoff of the player at site x $[u_x(s_x)]$ comes from five group interactions in which the player uses the same strategy ($s_x = A$ or B). Each group is formed by a focal player and their four nearest neighbors, as was considered in many previous evolutionary public goods games [47–49].

Using the previous notations for $n = 5$, the payoff parameters within a partnership group interaction are defined as

$$a_i = b_i = \begin{cases} 1 & \text{if } i = 2 \text{ or } 3 \\ 0 & \text{otherwise} \end{cases}, \tag{4}$$

where $i = 0, 1, ..., n$. Notice that the roles of strategy A and B are interchangeable. This game has $2^5 = 32$ microscopic states, 20 of which provide all players with a unit payoff; otherwise, the players receive no payoff. The former, finite-payoff states, however, are not strict Nash equilibria, since one of the three players with identical strategies can reverse their choice without modifying the payoff of any of the players. In this group interaction, the expected average payoff is $20/32 = 0.625$ if the players select their strategy at random. This setup is reminiscent of ice-type or six-vertex models [50,51], which can be solved exactly on a square lattice [52–54], in that they also define the energy via a restricted number of possible contributions based on the configuration of the nearest-neighbor bonds of each site.

The evolution of the spatial strategy arrangement is controlled by consecutive strategy updates that allow a randomly selected player (at site x) to modify their strategy unilaterally from s_x to s'_x with a probability exponentially favoring higher individual incomes. More precisely, this logit rule [24,33,34] defines the probability of a strategy update as

$$w(s_x \to s'_x) = \frac{e^{u_x(s'_x)/K}}{[e^{u_x(A)/K} + e^{u_x(B)/K}]}, \tag{5}$$

where K quantifies the amount of noise influencing the decision processes, and $u_x(s'_x)$, $u_x(A)$, and $u_x(B)$ are the payoffs player x receives when playing strategy s'_x, A, and B, respectively. For the kinetic Ising model [55], a similar dynamical rule drives the spin system toward thermodynamic equilibrium characterized by the Boltzmann distribution at temperature K. Thermodynamic behavior (including order-disorder phase transitions, critical phenomena, etc.) was observed in many evolutionary games in which two-payer n-strategy potential games define the pair interactions (for further references, see Ref. [27]). To compare these results, however, we have to take into consideration the fact that in physical systems the Hamiltonian summarizes the contributions of pair interactions (and not how, if at all, they are shared between the interacting particles) while the game theoretical potential summarizes the variations of individual payoffs that, in general, may or may not be equally beneficial for all interacting participants. In short, the group size n and possibly other details of the interaction should be taken into consideration when comparing noise and temperature dependence.

The selection of the group interaction given by Eq. (4) was originally motivated by a desire to create a horizontally (or vertically) oriented striplike arrangement of strategies A and B on the square lattice in ground states [see Fig. 2(a)], as the group formed by the nearest neighbors of any focal player in such an ordered arrangement maximize the payoff and the potential in this interaction. It turned out, however, that this group interaction also supports the formation of many other, tilted fibrous strategy arrangements. For example, Fig. 2(b) shows another translation invariant structure,

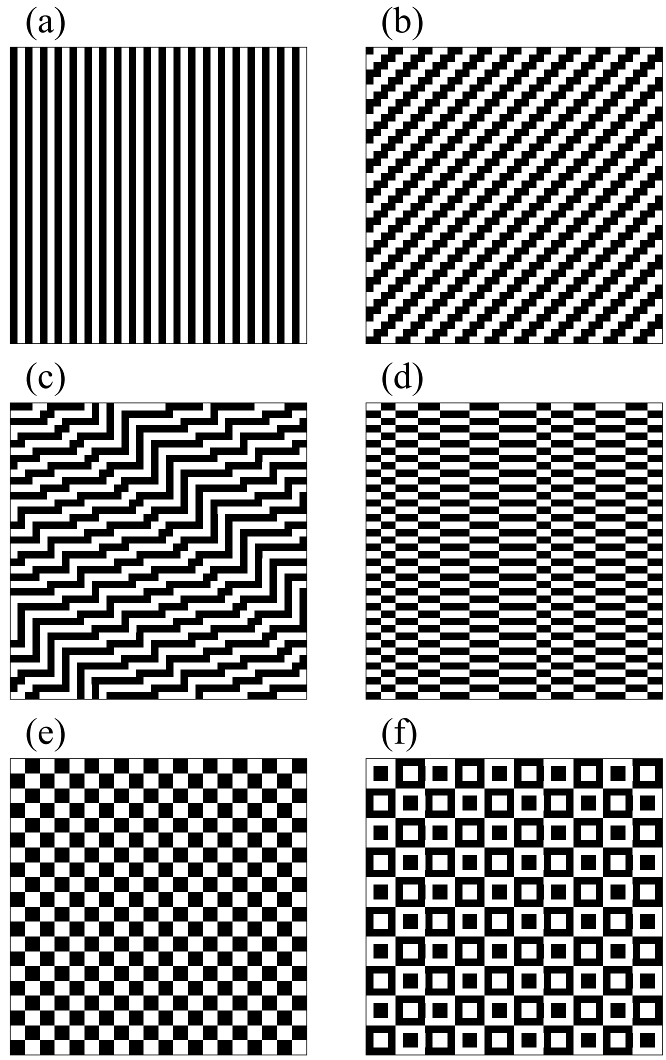

FIG. 2. Optimal spatial arrangements of two strategies for group interactions represented by white and black boxes corresponding to strategy A and B, respectively. Further fibrous strategy arrangements may be created via exploiting symmetries (e.g., translation, rotation, reflection, and strategy exchange, i.e., $A \leftrightarrow B$).

while Fig. 2(c) exemplifies a partially ordered optimal strategy structure. These spatial arrangements share a common feature: Strategies A and B represent equivalent roads of unit width that form closed loops on the surface of a torus. This topological constraint limits the number of states that realize this type of optimal arrangement.

Figure 2(d) also shows a partially ordered arrangement composed of parallel road segments with different lengths. Note the absence of closed loops. In contrast, the chessboardlike arrangement Fig. 2(e) is formed by 2×2 blocks of strategies A and B, which can be considered as periodically arranged four-edge loops. These four-edge loops, however, can be surrounded by another opposite square loop, and this expanded arrangement can be repeated many times, too. Figure 2(f) illustrates that one can build further periodic structures from such identical concentric loops for suitable values of L. Evidently, many other optimal strategy arrangements can be constructed in which two or three participants choose

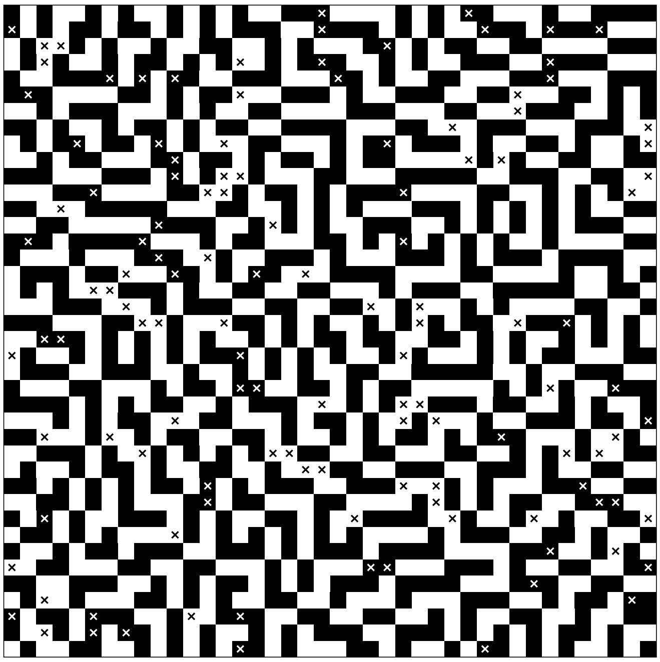

FIG. 3. Spatial arrangements of strategies in the stationary state at low noise level $K = 0.1$ in a section of 40×40 sites of a large system ($L = 1000$) after 10^6 MCS. The system was started from a random initial state. $\times$s with opposite color inside the black and white boxes indicate frustrated sites, that is, sites at which a strategy reversal results in no payoff change for the active player.

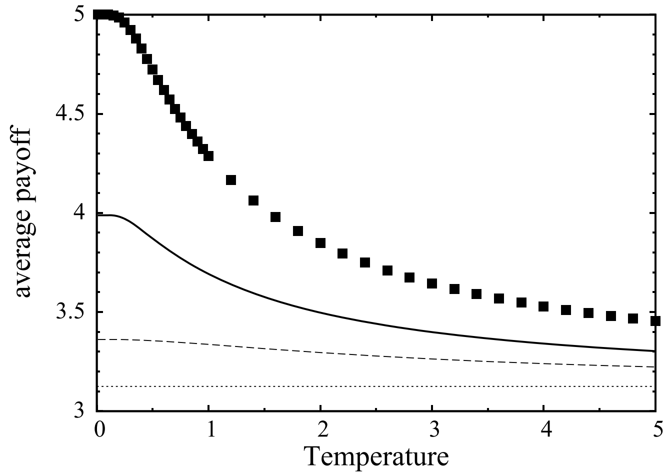

FIG. 4. Average payoff versus noise. The MC data (filled boxes) were obtained for $L = 800$, $\tau_s = 10^5$, and $\tau_r = 10^4$ MCS in the $K > 0.1$ region. For lower noise levels, a larger system size ($L = 1400$) and a longer thermalization time ($\tau_r > 10^5$ MCS) were used in order to mitigate finite-size and slowing-down effects that typically accompany the increase of the correlation length in ordering processes. The dotted, dashed, and solid lines show the predictions of the mean-field, pair-, and four-site approximations.

strategy A (or B) within each five-player neighbor group: One way to find arrangements like the ones mentioned above is to either come up with or observe in simulations (see the next section) compatible local strategy patterns that can be placed next to each other without lowering maximized player payoffs, such as the road segments highlighted in the previous descriptions. We have to emphasize that all strategy arrangements in Fig. 2 represent stable ground states (i.e., strict Nash equilibria) because any unilateral strategy reversal would result in a payoff decrease for all players in one or more groups that include the active player.

IV. RESULTS

To quantitatively analyze the macroscopic behavior of this spatial evolutionary game, we performed Monte Carlo (MC) simulations (see Supplemental Material [56]) on large lattices ($L \geqslant 800$) after a sufficiently long thermalization time (τ_r). The simulation time was measured in MC steps (MCS), the time it takes for each player to have a chance to modify their strategy once on average. The MC simulations indicated that τ_r increases significantly when K is decreased. More quantitatively, $\tau_r < 10^3$ MCS if $K > 0.15$, and $\tau_r > 10^5$ MCS if $K < 0.1$.

First, we show a typical strategy arrangement in the stationary state at a low noise level ($K = 0.1$), when the average payoff is 4.99990(2) in the stationary state. In spite of the very small number of unsatisfied players, the observed spatial snapshot of the strategy distribution (see Fig. 3) resembles a labyrinth (similar to those reported in [38,57–59]) rather than one of the fibrous optimal structures shown in Fig. 2. Additionally, the visualization of the time-dependent strategy distribution indicated clearly that the typical strategy reversals occur at the frustrated sites (sites at which a strategy reversal results in no payoff change for the active player) denoted by $\times$ symbols in the snapshot. Strategy reversals at solitary frustrated sites can be repeated many times. Within a group of neighboring frustrated sites, however, an individual local strategy change can relieve or introduce frustrations within a small neighborhood of the active player. Consequently, the consecutive complex strategy reversals at frustrated sites can modify their spatial patterns and also their density via a complex branching-annihilating phenomenon.

Another characteristic feature of these spatial structures is related to the interchangeability of strategies A and B: They occur with the same probability at the sites of the lattice for any value of K.

In order to have a quantitative picture about the macroscopic behavior, we determined the average payoff (Fig. 4) and the frequency of frustrated sites (Fig. 5) over a sampling time τ_s in the stationary state for different noise levels. Figure 4 shows that the average payoff decreases monotonously from its maximum value (5) and goes to the mentioned combinatorial prediction ($5 \times 20/32 = 3.125$) as $K \to \infty$ and the local correlations are suppressed by the noise. This latter value also coincides with the prediction of the mean-field approximation, shown by the dotted line in Fig. 4. In agreement with expectations, this approach predicts the two strategies to be present in equal frequency, that is, $\rho_A = \rho_B = 0.5$.

The effect of local correlations on the spatial strategy distributions can be investigated using dynamical cluster methods (details are given in [34,60–62]). These standard methods generalize the mean-field approximation by estimating the probabilities of possible configurations on clusters of multiple, in the present case two or four, sites via numerical

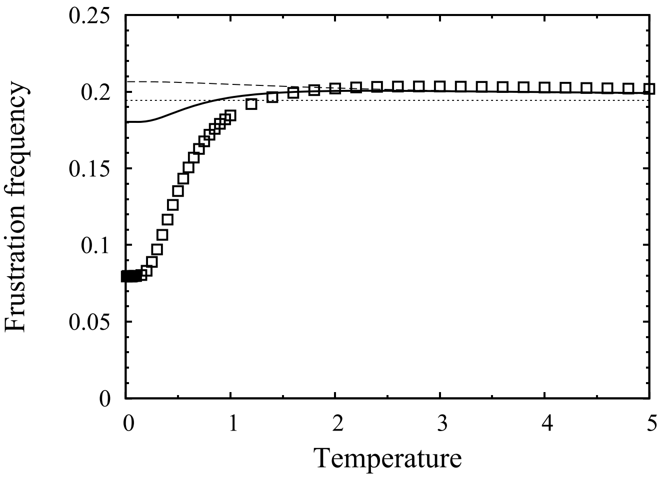

FIG. 5. Noise dependence of the average frequency of frustrated sites. The parameters of the MC simulations and line types used to show the results of the approximative methods are given in the caption of Fig. 4.

integration of a set of differential equations derived from the dynamical strategy update rule.

These approaches also predict $\rho_A = \rho_B = 0.5$ and the average payoff going to 3.125 while the spatial distribution of the strategies becomes entirely random as $K \to \infty$. At the same time, these methods cannot reproduce the low-noise behavior observed in the Monte Carlo simulations, which clearly indicates the presence of more complex local correlations.

The MC simulations show that the average frequency of frustrated sites has a wide local maximum at $K = 2.9(1)$ and goes to 0.19433(1) in the $K \to \infty$ limit, in general agreement with the predictions of the approximative analytical methods. In the opposite, $K \to 0$ limit, this quantity reaches its minimum value [0.0795(3)]. It is worth noting that the cluster variation methods cannot reproduce the extent of the decrease found in the frequency of frustrated sites via MC simulations. (See the low-temperature, $K < 1$, regime of Fig. 5.) We conjecture that this shortcoming of these approximative analytical methods is related to the abovementioned interplay between the spatial distribution of the strategies and frustration. This phenomenon also seems to cause extremely slow relaxation ($\tau_r > 10^6$ MCS) toward the stationary states at low noises ($K < 0.1$).

Finally, let us underline the absence of spontaneous symmetry breaking from our results, which we would typically expect to occur in spatial systems with just a few ordered ground states. In general, ordering processes in spatial thermodynamic game systems are initiated by the formation of clusters that maximize the potential of the local (be it pair or group) interactions of all involved sites. Usually there are only a few different strategy arrangements capable of this, which are related to each other by the symmetries of the interaction; for example, the two kinds of coordinated domains in the Ising model [7–9], which can be transformed into each other by flipping all constituent spins. When the temperature is low enough, this nucleation process can develop a number of larger domains of each locally maximizing arrangement, which keep growing (increasing the correlation length) until one of them eventually percolates and then spreads through the whole system, displacing the other, symmetry-related arrangements. Here, in the present group interaction model, no such symmetry breaking seems to take place. In particular, the four-site configuration probabilities confirm the presence of rotation and reflection symmetries. This is probably caused by the high number of potential maximizing local strategy arrangements and their high degree of compatibility, which allows them to touch or overlap without lowering the potential.

V. SUMMARY

In the present work we proved the existence of a potential in evolutionary games in which a generalized group interaction of n players [22,23] controls the evolutionary dynamics. In these two-strategy group interactions, the existence of a potential is intimately related to the equivalence of the players: Those of them who choose identical strategies receive equal payoffs that are determined by the number of the players that selected each strategy. We exploited the simple topological structure of the dynamical graph (an n-dimensional hypercube) that becomes visible on a suitably chosen two-dimensional projection to evaluate the potential. In the light of the present results, we can imagine that games with different sets of payoffs that conform to other (possibly orthogonal) two-dimensional projections of the hypercube representing the dynamical graph [63] may also admit a similarly easily determinable potential.

To explore what kinds of phenomena may occur in spatial systems with this type of group interaction, we investigated a very simple example supporting the formation of a wide range of strategy distributions on a square lattice. In the low noise limit, the MC simulations indicated the dominance of labyrinthlike, slowly evolving strategy arrangements similar to those observed in oscillatory media [58,59]. In these chemical-reaction systems, however, transitions are observed between the topologically different patterns when the parameters are tuned. We think that a modified version of this group interaction may allow us to study similar transitions using the Ising formalism.

MC simulations and analytical approximations demonstrated the relevant effects of the temperature on both the stationary states and the relaxation processes. In the low noise limit the players receive the maximal payoff, while the resultant strategy arrangements contain frustrated sites. The interplay between the large number of optimal (ground) states and frustrated sites causes extremely slow evolution toward the stationary state, similar to the slowing down found in spin glasses [64] or other frustrated spin systems [65,66]. Different variants of this model could possibly be used to clarify mechanisms underlying similar phenomena in ice-type models [50,51] including metamaterials [67–70] and other evolutionary games.

DATA AVAILABILITY

The data that support the findings of this article are not publicly available upon publication because it is not technically feasible and/or the cost of preparing, depositing, and hosting the data would be prohibitive within the terms of this research project. The data are available from the authors upon reasonable request.

[1] W. Weidlich, Br. J. Math. Psychol. **24**, 251 (1971).
[2] M. A. Nowak and R. M. May, Nature (London) **359**, 826 (1992).
[3] C. Castellano, S. Fortunato, and V. Loreto, Rev. Mod. Phys. **81**, 591 (2009).
[4] J. M. Smith and G. R. Price, Nature (London) **246**, 15 (1973).
[5] J. M. Smith, *Evolution and the Theory of Games* (Cambridge University Press, Cambridge, UK, 1982).
[6] J. Hofbauer and K. Sigmund, *The Theory of Evolution and Dynamical Systems* (Cambridge University Press, Cambridge, UK, 1988).
[7] S. G. Brush, Rev. Mod. Phys. **39**, 883 (1967).
[8] C. Domb, in *Phase Transitions and Critical Phenomena*, Vol. 3, edited by C. Domb and M. S. Green (Academic Press, London, 1974), pp. 357–484.
[9] D. Sornette, Rep. Prog. Phys. **77**, 062001 (2014).
[10] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behaviour* (Princeton University Press, Princeton, NJ, 1944).
[11] G. Hardin, Science **162**, 1243 (1968).
[12] A. J. C. van de Kragt, J. M. Orbell, and R. M. Dawes, Am. Polit. Sci. Rev. **77**, 112 (1983).
[13] R. M. Isaac and J. M. Walker, Quart. J. Econ. **103**, 179 (1988).
[14] K. G. Binmore, *Game Theory and the Social Contract: Playing Fair* (MIT Press, Cambridge, MA, 1994).
[15] E. Fehr and S. Gächter, Am. Econ. Rev. **90**, 980 (2000).
[16] H. Brandt, C. Hauert, and K. Sigmund, Proc. R. Soc. Lond. B **270**, 1099 (2003).
[17] F. Dionisio and I. Gordo, Evol. Ecol. Res. **8**, 321 (2006).
[18] J. Y. Wakano, Math. Biosci. **201**, 72 (2006).
[19] M. Archetti, J. Evol. Biol. **22**, 2192 (2009).
[20] T. L. Czárán and R. F. Hoekstra, PLoS ONE **4**, e6655 (2009).
[21] R. Hardin, Syst. Res. **16**, 472 (1971).
[22] M. Broom, C. Cannings, and G. T. Vickers, Bull. Math. Biol. **59**, 931 (1997).
[23] A. Li, M. Broom, J. Du, and L. Wang, Phys. Rev. E **93**, 022407 (2016).
[24] L. E. Blume, Games Econ. Behav. **5**, 387 (1993).
[25] D. Monderer and L. S. Shapley, Games Econ. Behav. **14**, 124 (1996).
[26] O. Candogan, I. Menache, A. Ozdaglar, and P. A. Parrilo, Math. Oper. Res. **36**, 474 (2011).
[27] G. Szabó and I. Borsos, Phys. Rep. **624**, 1 (2016).
[28] C. B. Cadsby and E. Maynes, J. Econ. Behav. Org. **34**, 603 (1998).
[29] L. A. Bach, T. Helvik, and F. B. Christiansen, J. Theor. Biol. **238**, 426 (2006).
[30] J. Wang, F. Fu, T. Wu, and L. Wang, Phys. Rev. E **80**, 016101 (2009).
[31] J. M. Pacheco, F. C. Santos, M. O. Souza, and B. Skyrms, Proc. Biol. Sci. **276**, 315 (2009).
[32] M. Perc, J. Gómez-Gardeñes, A. Szolnoki, L. M. Floría, and Y. Moreno, J. R. Soc. Interface **10**, 20120997 (2013).
[33] D. Fudenberg and D. Levine, *The Theory of Learning in Games* (MIT Press, Cambridge, MA, 1998).
[34] G. Szabó and G. Fáth, Phys. Rep. **446**, 97 (2007).
[35] W. B. Arthur, Am. Econ. Rev. **84**, 406 (1994).
[36] D. Challet and M. Marsili, Phys. Rev. E **60**, R6271(R) (1999).
[37] D. Challet, M. Marsili, and Y. Zhang, *Minority Games: Interacting Agents in Financial Markets* (Oxford University Press, Oxford, 2004).
[38] E. Burgos, H. Ceva, and J. P. R. Perazzo, Physica A **337**, 635 (2004).
[39] J. Ashkin and E. Teller, Phys. Rev. **64**, 178 (1943).
[40] P. W. Anderson, Mater. Res. Bull. **8**, 153 (1973).
[41] J. M. Kosterlitz and D. J. Thouless, J. Phys. C: Solid State Phys. **6**, 1181 (1973).
[42] G. Parisi, Phys. Rev. Lett. **50**, 1946 (1983).
[43] S. Kivelson and S. Sondhi, Nat. Rev. Phys. **5**, 368 (2023).
[44] H. T. Diep, C. R. Phys. **26**, 225 (2025).
[45] J. Schnakenberg, Rev. Mod. Phys. **48**, 571 (1976).
[46] J. Hofbauer and K. Sigmund, *Evolutionary Games and Population Dynamics* (Cambridge University Press, Cambridge, UK, 1998).
[47] C. Hauert, S. De Monte, J. Hofbauer, and K. Sigmund, Science **296**, 1129 (2002).
[48] G. Szabó and C. Hauert, Phys. Rev. Lett. **89**, 118101 (2002).
[49] A. Szolnoki, M. Perc, and G. Szabó, Phys. Rev. E **80**, 056109 (2009).
[50] L. Pauling, J. Am. Chem. Soc. **57**, 2680 (1935).
[51] J. C. Slater, J. Chem. Phys. **9**, 16 (1941).
[52] E. H. Lieb, Phys. Rev. **162**, 162 (1967).
[53] B. Sutherland, Phys. Rev. Lett. **19**, 103 (1967).
[54] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic Press, London, 1982).
[55] R. J. Glauber, J. Math. Phys **4**, 294 (1963).
[56] See Supplemental Material at https://link.aps.org/supplemental/10.1103/nhxn-6zqf for the C code used to perform the Monte Carlo simulations analyzed in the present paper.
[57] K. Shtengel and L. P. Chayes, J. Stat. Mech. (2005) P07006.
[58] V. K. Vanag and I. R. Epstein, Int. J. Dev. Biol. **53**, 673 (2009).
[59] C. Huang, Z. Song, and Z. Qu, Phys. Rev. E **111**, L032201 (2025).
[60] H. A. Gutowitz, J. D. Victor, and B. W. Knight, Physica D **28**, 18 (1987).
[61] R. Dickman, Phys. Rev. A **38**, 2588 (1988).
[62] G. Szabó, A. Szolnoki, and R. Izsák, J. Phys. A: Math. Gen. **37**, 2599 (2004).
[63] F. Harary, Comput. Math. Applic. **15**, 277 (1988).
[64] G. Parisi, Phys. Rev. Lett. **43**, 1754 (1979).
[65] G. Toulouse, Commun. Phys. **2**, 115 (1977).
[66] H. T. Diep, *Frustrated Spin Systems* (World Scientific, Singapore, 2004).
[67] M. J. Morrison, T. R. Nelson, and C. Nisoli, New J. Phys. **15**, 045009 (2013).
[68] C. Nisoli, R. Moessner, and P. Schiffer, Rev. Mod. Phys. **85**, 1473 (2013).
[69] R. van Mastrigt, C. Coulais, and M. van Hecke, Phys. Rev. E **108**, 065002 (2023).
[70] C. Sirote-Katz, O. Palti, N. Spiro, T. Kálmán, and Y. Shokef, Phys. Rev. Res. **7**, 033004 (2025).