

**Evolutionary matching-pennies game on bipartite regular networks**György Szabó,<sup>1</sup> Levente Varga,<sup>1,2</sup> and István Borsos<sup>1</sup><sup>1</sup>*Institute of Technical Physics and Materials Science, Research Centre for Natural Sciences, Hungarian Academy of Sciences, P.O. Box 49, H-1525 Budapest, Hungary*<sup>2</sup>*Babeş-Bolyai University, RO-40084 Cluj-Napoca, Romania*

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Evolutionary games are studied here with two types of players located on a chessboard or on a bipartite random regular graph. Each player's income comes from matching-pennies games played with the four neighbors. The players can modify their own strategies according to a myopic strategy update resembling the Glauber dynamics for the kinetic Ising model. This dynamical rule drives the system into a stationary state where the two strategies are present with the same probability without correlations between the nearest neighbors while a weak correlation is induced between the second and the third neighbors. In stationary states, the deviation from the detailed balance is quantified by the evaluation of entropy production. Finally, our analysis is extended to evolutionary games where the uniform pair interactions are composed of an anticoordination game and a weak matching-pennies game. This system preserves the Ising type order-disorder transitions at a critical noise level decreasing with the strength of the matching-pennies component for both networks.

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**I. INTRODUCTION**

Evolutionary games give us a general mathematical background for the investigation of multiagent social and biological systems even in the cases where the pair interactions do not possess the symmetries characteristic to many-particle physical systems. For the simplest systems, the pair interaction is described by two-player two-strategy games. Among these so-called  $2 \times 2$  matrix games, the matching-pennies game represents an interaction that is missing in physical systems. Despite it, the effect of this type of interaction can be well investigated with the concepts and tools of nonequilibrium statistical physics. The successes of these approaches are well documented by the large number of papers published in the literature of physics (for a general survey and references, see the book [1] and review [2]).

In the multiagent evolutionary games, the players are distributed on a lattice or network, and their interactions with the connected neighbors are described by the payoff matrix introduced in traditional game theory [3]. In these models, each player chooses one of her possible strategies in all the games in which she participates. For most of the cases, we assume that the players, their strategy sets, and the games are identical. Furthermore, for the evolutionary games, the players can modify their own strategies with following a (uniform) dynamical rule that may be deterministic or stochastic, synchronized or random sequential update, based on imitation of a more successful neighbor or myopic selection of those strategies providing higher individual gain for the given surrounding.

In the present paper, we restrict our analysis to those evolutionary games where two types of players ( $X$  and  $Y$ ) are distinguished who are located on bipartite networks. More precisely, first we study these evolutionary games on a square lattice that can be divided into two equivalent sublattices (using the analogy of the white and black squares on a chessboard) occupied by players of types  $X$  and  $Y$ , respectively. Additionally, we will study this model on a bipartite random regular graph where each player of type  $X$

interacts with four connected neighbors of type  $Y$  as happens on a chessboard.

In order to clarify the effects of matching-pennies interactions, first we study spatial distribution of the two choices called heads ( $H$ ) and tails ( $T$ ). For the traditional matching-pennies game, the two players first agree who will be the winner if the sides of the coins are the same or different (henceforth, representing the types  $X$  and  $Y$ ). Then they conceal a coin in their palms with the side heads or tails upward and reveal their choices simultaneously. The winner receives the opponent's penny. It is evidently a zero-sum game, and the outcomes with the suitable payoffs are illustrated within the boxes in the flow diagram plotted in Fig. 1. In this flow diagram, the boxes illustrate the four possible strategy pairs (called strategy profiles), and the arrows on the horizontal and vertical edges point toward the strategy profile where the strategy reversing player receives a higher individual payoff. The directed loop indicates that the strategy profile changes cyclically if the players are allowed to modify their own strategy alternately or even in a random sequential order. This inherent feature of the matching-pennies game can be considered as a driving force creating a constant probability current along the directed loop in the stationary state.

Here it is worth mentioning that the matching-pennies game, as a driving force for the Red Queen mechanism, was proposed by van Valen [4,5] in the years when the concept of evolutionary game theory emerged [6]. Similar interaction was suggested to describe the competition between buyers and sellers [7] or property owners and criminals [8].

In the present paper, we study the effect of the matching-pennies game on the strategy distribution for two types of networks when varying the noise level at a Glauber type dynamical rule. It will be demonstrated that this interaction develops a random strategy distribution with a weak correlation for the spatial connectivity structure while this correlation changes slightly on the bipartite random regular graph. In the stationary state, the most striking feature of the matching-pennies game is related to the maintenance of a cyclic strategy variation that breaks the detailed balance. The

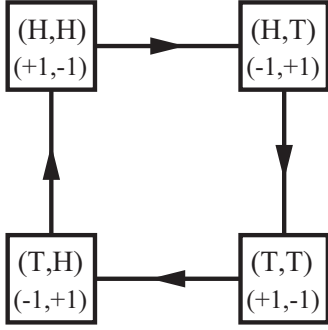


FIG. 1. Flow graph for the two-player matching-pennies game.

strength of the latter effect will be quantified by the evaluation of the entropy production. Subsequently, we will consider the effect of a weak matching-pennies game on the order-disorder transition when the chessboardlike arrangement of strategies is supported by antcoordination games (or antiferromagnetic interactions) at low noise levels. The relevance of these models is justified by the strong analogy between the Ising models [9,10] and the social [11–14] or [15] biological systems. It will be demonstrated that the main features of this critical transition remain unchanged, whereas, the critical noise level decreases with the strength of the matching-pennies component.

## II. MODELS, FORMALISM, AND METHODS

We study multiagent evolutionary games with players located on the sites of bipartite regular graphs. These structures can be divided into two sets (denoted as  $\mathcal{X}$  and  $\mathcal{Y}$  referring to the types of players) of nodes  $x \in \mathcal{X}$  and  $y \in \mathcal{Y}$ . The analysis is restricted to networks where each site  $x$  (and  $y$ ) is connected to  $k = 4$  neighboring sites denoted as  $x + \delta \in \mathcal{Y}$  (or  $y + \delta \in \mathcal{X}$ ). Each player's income comes from two-player matrix games with the neighbors. Following the traditional notation, the strategy for both types of players is denoted by two-dimensional unit vectors as

$$\mathbf{s}_x, \mathbf{s}_y = H = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad T = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \quad (1)$$

For this formalism, the player's income is expressed by a sum of matrix products,

$$u_x = \sum_{\delta} \mathbf{s}_x \cdot \mathbf{A} \mathbf{s}_{x+\delta}, \quad u_y = \sum_{\delta} \mathbf{s}_y \cdot \mathbf{B} \mathbf{s}_{y+\delta}, \quad (2)$$

where the summation runs over the four nearest neighbor sites belonging to the opposite sublattice. The matrix elements  $A_{ij}$  and  $B_{ij}$  ( $i, j = 1, 2$ ) tabulate the payoffs for players of types  $X$  and  $Y$ . For the matching-pennies game, the payoff matrices are defined as

$$\mathbf{A} = \begin{pmatrix} +1 & -1 \\ -1 & +1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} -1 & +1 \\ +1 & -1 \end{pmatrix}. \quad (3)$$

It is known that this zero-sum game has only one mixed Nash equilibrium when both players choose the two strategies ( $H$  and  $T$ ) with the same probability ( $1/2$ ) independent of each other.

We also study systems when the payoffs are composed of antcoordination and matching-pennies games as

$$\mathbf{A} = \begin{pmatrix} 0 + \epsilon & 1 - \epsilon \\ 1 - \epsilon & 0 + \epsilon \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 - \epsilon & 1 + \epsilon \\ 1 + \epsilon & 0 - \epsilon \end{pmatrix}. \quad (4)$$

Contrary to the matching-pennies game, the antcoordination game [with payoff matrices (4) for  $\epsilon = 0$ ] has two pure Nash equilibria when  $s_x = H$  and  $s_y = T$  or  $s_x = T$  and  $s_y = H$ . For the spatial multiagent model, the corresponding strategy arrangements are equivalent to the chessboard or antichessboard spatial arrangement of the two strategies. Within the terminology of bipartite graphs, we can say that sublattices  $\mathcal{X}$  and  $\mathcal{Y}$  are occupied uniformly by opposite strategies, that is,  $s_x = H$  and  $s_y = T$  (or  $s_x = T$  and  $s_y = H$ )  $\forall x \in \mathcal{X}$  and  $\forall y \in \mathcal{Y}$ .

Henceforth, we assume that the evolution of the strategy distribution is governed by the same random, sequential, and stochastic strategy reversals (e.g.,  $s_x \rightarrow s'_x$ ) on both sublattices. Within an elementary step, a randomly chosen single player (e.g., at site  $x$ ) modifies her strategy with a probability,

$$W(\mathbf{s}_x \rightarrow \mathbf{s}'_x, \mathbf{s}_{-x}) = \frac{1}{1 + \exp[(u_x - u'_x)/K]}, \quad (5)$$

dependent on her payoff variation, whereas, the strategy profile for the rest of players (denoted traditionally as  $\mathbf{s}_{-x}$ ) remains unchanged. This strategy modification is similar to the Glauber dynamics [16] used in the kinetic Ising model and favors the payoff increase for the given player, and  $K$  characterizes the magnitude of noise.

It is already well known that this transition rule [2] and the logit rules [17,18] drive evolutionary games into a Boltzmann distribution if the payoff matrices are equivalent and symmetric (i.e.,  $\mathbf{A} = \mathbf{B} = \mathbf{A}^T$ ) or if we can derive a potential matrix [19,20] summarizing the individual gain of the player modifying her strategy. Consequently, for the pair interactions given by the matrices (4) at  $\epsilon = 0$ , the system becomes equivalent to the antiferromagnetic Ising model without a magnetic field that exhibits a continuous phase transition from the antiferromagnetic state to the paramagnetic phase at a critical point if the temperature (noise) is increased. The second model [defined by payoffs (4) for  $0 < |\epsilon| \ll 1$ ] allows us to study how the presence of a weak matching-pennies component affects the order-disorder transition.

The above systems are analyzed by performing Monte Carlo (MC) simulations on square lattices of  $N = L \times L$  sites with periodic boundary conditions and bipartite random regular graphs (of site  $N$ ) generated by an algorithm prohibiting the presence of double edges. During the simulations,  $N$  is varied from  $N = 2.5 \times 10^5$  to  $4 \times 10^6$ . For most of the cases, the simulations are started from a random initial state, and after a suitable thermalization time  $t_{\text{th}}$ , the statistical data are obtained by averaging over a sampling time  $t_s$ . These values are adjusted to the system behavior and are changed typically from  $t_{\text{th}} = 2000$  Monte Carlo steps (MCSs) and  $t_s = 50\,000$  MCSs to  $t_{\text{th}} = 10^6$  MCSs and  $t_s = 3 \times 10^6$  MCSs where, within the time unit (MCS), each site has a chance, once on average, to modify its state. The larger sizes and longer times are used when we have to achieve high accuracy (e.g., for the determination of correlations) or in the vicinity of the critical transitions to suppress the undesired effects of fluctuations.

Using MC simulations, we have determined the correlation functions, the entropy production, the order parameter, and the fluctuations as a function of noise  $K$ .

Besides the MC simulations, we have used the so-called dynamical cluster methods to obtain approximate analytical results. Here we do not wish to describe the details of these calculations as the well-documented dynamical cluster methods (frequently called generalized mean-field approximations) are already applied successfully for many other systems [21,22], including evolutionary games [2,23]. For these methods, we evaluate all the configuration probabilities on a compact cluster built up from  $n$  sites on translation invariant lattices. Evidently, increasing  $n$  improves the accuracy and the number of variables we have to evaluate when applying this approximate method [24]. For example, at the nine-site level of these approximations, we have to numerically solve a set of equations of motion where the  $2 \times 2^9 = 1024$  variables describe the probability for each possible strategy configuration on the two distinguishable  $3 \times 3$  blocks of sites. With the knowledge of these configuration probabilities, we can give analytical predictions for the above-mentioned quantities.

### III. CORRELATIONS FOR THE MATCHING-PENNIES GAMES

According to the MC simulations, the system with the matching-pennies interactions evolves into a stationary state where both strategies in both sublattices are present with the same probability. These results are in accordance with symmetries predicting equivalence between strategies  $H$  and  $T$  and sublattices  $X$  and  $Y$ . When displaying the strategy distribution on the square lattice, we cannot visually observe correlations.

In order to get a more quantitative picture, we have determined the one-time correlation functions defined as

$$c(\mathbf{z}, t) = \langle n_H(\mathbf{x}, t) n_H(\mathbf{x} + \mathbf{z}, t) \rangle_x - \langle n_H(\mathbf{x}, t) \rangle_x^2 \quad (6)$$

among the first [ $\mathbf{z} = (1, 0)$ ], second [ $\mathbf{z} = (1, 1)$ ], and third neighbor [ $\mathbf{z} = (2, 0)$ ] sites, where  $\langle \cdots \rangle_x$  denotes the average over the whole lattice and the occupation numbers  $n_H(\mathbf{x}, t) = 1$  if  $\mathbf{s}_x = H$  and zero otherwise at time  $t$ . In the stationary state,  $c(\mathbf{z}, t)$  fluctuates around its mean value [denoted shortly as  $c(\mathbf{z})$  and determined by averaging over a sufficiently long time period] with a magnitude dependent on size  $N$  and noise  $K$ . Due to the symmetries, the same average correlations are expected if we consider the distribution of the  $T$  strategies. It is noteworthy that we do not need to distinguish the sublattices.

The MC simulations have quantitatively confirmed the absence of correlations between the nearest neighbor sites for any values of  $K$  as indicated in Fig. 2 where the statistical errors are comparable with the symbol size. On the other hand, weak negative correlations are found between the second and the third neighbors. Evidently, the magnitude of these correlations vanishes if  $K \rightarrow \infty$ .

The two-site (pair) approximation has predicted no correlations, that is, all the possible two-site configurations appeared with the same probability. Similar results are obtained for the four-site approximation when  $2^4 = 16$  configuration probabilities are determined on a  $2 \times 2$  block of sites on the square lattice. The failure of the latter approach has motivated

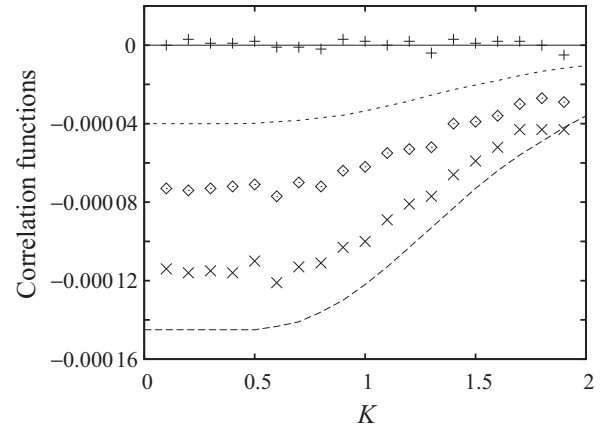


FIG. 2. Average correlation functions vs  $K$  for the matching-pennies games on a chessboard. Pluses illustrate the absence of correlation between the nearest neighbor sites; crosses (diamonds) show  $c(\delta)$  for the second (third) neighbors, respectively. The solid, dashed, and dotted lines illustrate the theoretical predictions based on the nine-site dynamical cluster method for the correlations among the first, second, and third neighbors.

us to evaluate the nine-site configuration probabilities too. This level of the dynamical cluster methods has clearly indicated the differences in the configuration probabilities, meanwhile, all the relevant symmetries (rotation, reflection, and equivalence among the two strategies and sublattices) are clearly exhibited. From these data, we could determine the values of the correlation function for the same distances as above by averaging over a suitable set of configurations. The predictions are evaluated for different noise levels and are compared to the results of the MC simulations in Fig. 2. Notice the qualitative agreement between the results of the two methods. Evidently, one can achieve better agreement by performing this approach for higher levels (e.g., on  $4 \times 4$  clusters). Unfortunately, this time-consuming analysis exceeds the capacity of our computers.

As the above calculations have not clarified the effects of the topological features of the connectivity network on the emerging weak spatial correlations, therefore, the above analyses are repeated on bipartite random regular graphs for the same number of neighbors ( $k = 4$ ). Figure 3 illustrates the MC results on the bipartite random graphs that show some differences in comparison to those obtained on the square lattice (see Fig. 2). The most striking results are the smaller magnitudes of the correlations on the bipartite random regular graphs and their absence between the third neighbor sites.

Before discussing the above results, first we emphasize that, at the level of pair approximation, the square lattice cannot be distinguished from the Bethe lattice we assumed for the analytical calculations. The loop-free Bethe lattice can be considered as an adequate network locally similar to a sufficiently large random regular graph where the shortest loop size (involving a node) increases with the logarithm of size  $N$  [25]. Previous investigations [26] have clearly demonstrated the applicability of this approach for systems where the presence of sufficiently long loops has not modified

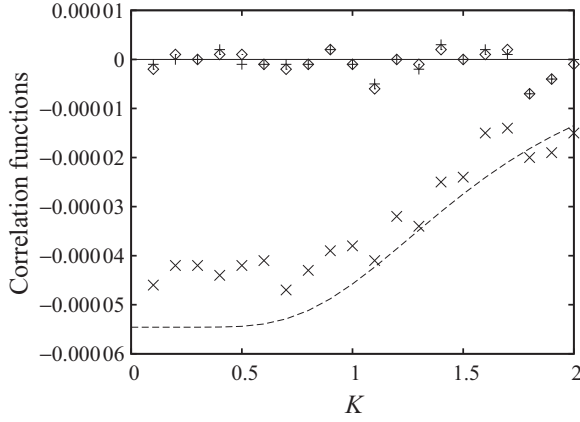


FIG. 3. Correlation functions vs  $K$  for the matching-pennies games on bipartite random regular graphs. Symbols illustrate the MC results as in Fig. 2. The solid and dashed lines show the approximative results extracted from the eight-site dynamical cluster method.

the macroscopic behavior of the system. The dynamical cluster method on a starlike five-site cluster has predicted a weak deviation from the uniform configuration probabilities. These differences, however, have not explained the presence of a weak correlation between the second neighbors found by Monte Carlo simulations for any noise levels. The more sophisticated eight-site dynamical cluster method, however, is capable of qualitatively reproducing the features found by MC simulations among the first, second, and third neighbors as illustrated by the solid and dashed lines in Fig. 3.

The differences between the correlation functions indicate the relevance of the short loops in the connectivity structures, although they are not the only source of the observed correlations. Additionally, we mention that significantly larger correlations are found in the zero noise limit if we block the frustrated transitions ( $u_x = u'_x$ ).

#### IV. BREAKING OF DETAILED BALANCE

Besides the weakly correlated spatial randomness, the breaking of detailed balance can also be quantified in the stationary states of the evolutionary matching-pennies games. For this purpose, we consider the entropy production (for a survey, see Ref. [27]). This quantity summarizes the contributions coming from the breaking of detailed balance between all the possible elementary transition pairs when only one player (e.g., at site  $x$ ) changes her strategy from  $s_x$  to  $s'_x$  and backward (from  $s'_x$  to  $s_x$ ) while all the other strategies are fixed. The frequency of this elementary transition from the strategy profile  $(s_x, s_{-x})$  to  $(s'_x, s_{-x})$  will be denoted as

$$w(s_x \rightarrow s'_x, s_{-x}) = f(s_x, s_{-x})W(s_x \rightarrow s'_x, s_{-x}), \quad (7)$$

where  $f(s_x, s_{-x})$  is the probability of finding the system in the given strategy profile in the stationary state and the probability of strategy reversal  $W(s_x \rightarrow s'_x, s_{-x})$  is given by (5). In such types of systems, the deviation from the detailed balance can be quantified by the evaluation of entropy production defined

as

$$I = \frac{1}{2} \sum_{\substack{s_x, s'_x, \\ s_{-x}}} [w(s_x \rightarrow s'_x, s_{-x}) - w(s'_x \rightarrow s_x, s_{-x})] \times \ln \frac{w(s_x \rightarrow s'_x, s_{-x})}{w(s'_x \rightarrow s_x, s_{-x})}, \quad (8)$$

where the summation runs over  $\forall s_x, s'_x, s_{-x}$ . Notice that this quantity is always positive ( $I > 0$ ) except in the case where the conditions of detailed balance are satisfied, namely, if  $w(s_x \rightarrow s'_x, s_{-x}) = w(s'_x \rightarrow s_x, s_{-x}) \forall s_x, s'_x, s_{-x}$ .

Within the concept of dynamical graphs [27], the nodes represent the possible strategy profiles. Edges connect those strategy profiles that can be transformed into each other by a strategy change for a single player. Along these edges, we can define a probability current (as a difference between the forward and the backward transition frequencies) that should satisfy the Kirchhoff laws [28] in the stationary state.

In lattice systems, the entropy production can easily be estimated if we recognize that the transition frequency  $w(s_x \rightarrow s'_x, s_{-x})$  depends dominantly on the close neighborhood of player  $x$  if the dynamics is controlled by short range interactions. Besides it, we can exploit the translation invariance of the system that means that all these transitions are independent of  $x$ . As a result, the specific entropy production ( $I/N$ ) can be well estimated by determining the average transition frequencies for all possible local strategy configurations only within a close neighborhood of site  $x$ . Accordingly, in MC simulations,  $I/N$  is determined by considering the transition frequencies  $w(s_x \rightarrow s'_x, s_{-x})$  as a function of the  $2^4 = 16$  possible strategy configurations on the four nearest neighbor sites. In general, a better approximation can be obtained if we distinguish  $2^8 = 256$  configurations on the first and second neighbor sites around the focal player. In this way, we can deduce two approximative results for the specific entropy production ( $I/N$ ) whose comparison can indicate the relevance of the second neighbors, although they do

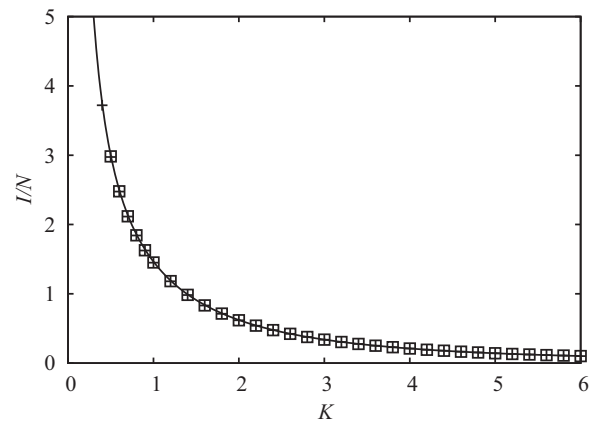


FIG. 4. Approximative results for the specific entropy production as a function  $K$  in the matching-pennies games on a chessboard. The MC data are denoted as symbols when only the first (pluses) and first and second neighbors (open boxes) are taken into consideration. The analytical result in the absence of correlations is illustrated by the solid line.

not directly influence the transition in the present model [29].

Figure 4 shows the MC results for the above-mentioned two approaches. Assuming the absence of correlations,  $I/N$  can easily be evaluated analytically. It is noteworthy that the nine-site approximation allows us to give a more accurate estimation for the specific entropy production. This calculation indicates that the weak correlation results in an extremely small deviation from the uncorrelated case (the difference is significantly less than the line thickness). In agreement with the expectations, the specific entropy production vanishes in the limit  $K \rightarrow \infty$  when the effect of payoffs is suppressed by the noise. On the other hand,  $I/N$  diverges when  $K \rightarrow 0$  due to the fact that some transitions become one way [e.g.,  $w(H \rightarrow T, s_{-x}) \rightarrow 0$  while  $w(T \rightarrow H, s_{-x}) \rightarrow c > 0$ ]. The analytical calculations in the absence of correlations show that the specific entropy production diverges as  $I/N \propto 1/K$  in the zero noise limit. On the contrary, this quantity vanishes as  $I/N \propto 1/K^2$  if  $K \rightarrow \infty$ .

### V. EVOLUTIONARY ANTICOORDINATION GAME WITH DISTURBANCE OF MATCHING PENNIES

Now we consider the second model when the ordering effect of the antcoordination type interactions is disturbed by a weak matching-pennies component [as defined by the payoff (4) for  $|\epsilon| \ll 1$ ] in the spatial evolutionary games. For the absence of the matching-pennies component ( $\epsilon = 0$ ), this model is equivalent to the well-known antiferromagnetic Ising model that has two equivalent ordered strategy arrangements in the zero noise limit as mentioned in the Introduction. For low noise levels, point defects will decorate the ordered structure, and these defects become larger and more frequent when  $K$  is increased. Above a critical noise level ( $K > K_c$ ), the strategy frequencies cannot be distinguished in the two sublattices. This order-disorder transition usually is quantified by the introduction of an order parameter  $\phi$  that is expressed by the difference in average occupation numbers,

$$m(t) = \langle n_H(\mathbf{x}, t) \rangle_x - \langle n_H(\mathbf{y}, t) \rangle_y, \quad (9)$$

as

$$\phi = |\langle m(t) \rangle_t|, \quad (10)$$

where  $\langle \dots \rangle_t$  denotes averaging over time in the stationary state. The noise dependence of the order parameter is well discussed in the literature of Ising models [10,30,31], and it represents a universal class of critical phase transitions occurring in many other systems. One of the main characteristics of these systems is that the order parameter exhibits power law behavior in the close vicinity of the critical point, namely,

$$\phi \propto (K_c - K)^\beta, \quad (11)$$

if  $0 < K_c - K \ll K_c$ . For the Ising model with nearest neighbor interactions on the square lattice, the values of the critical point and the exponent are known exactly. For the present parameters, the numerical values are  $K_c(\epsilon = 0) = 1/\ln(1 + \sqrt{2}) = 1.134593$  and  $\beta = 1/8$ .

Figure 5 compares the order parameters  $\phi$  as a function of  $K$  for  $\epsilon = 0$  and 0.2. Notice that, for  $\epsilon = 0$ , the MC data (closed boxes) coincide with the exact result found

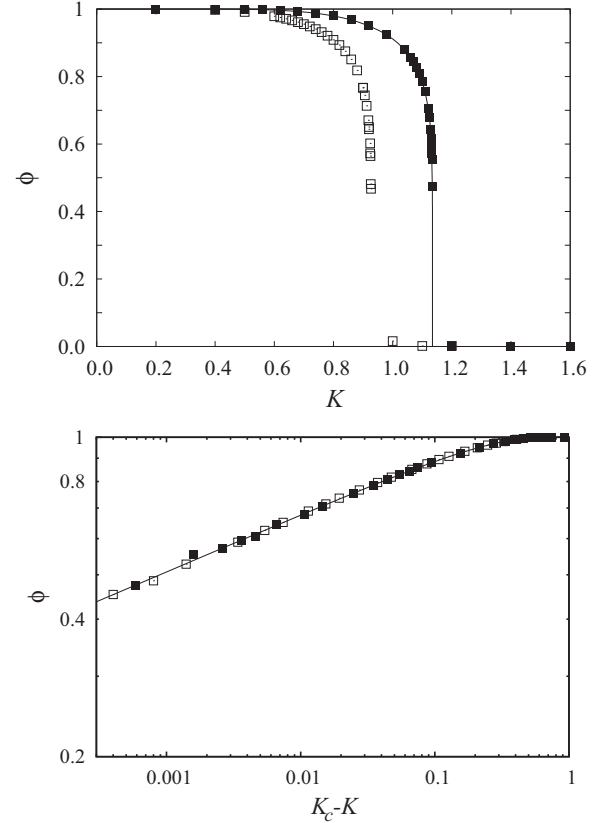


FIG. 5. MC results for the order parameter  $\phi$  as a function of  $K$  in the upper plot for the evolutionary antcoordination game on the square lattice in the absence (closed boxes) or presence (open boxes at  $\epsilon = 0.2$ ) of the matching-pennies component. The solid line shows the exact result for the corresponding antiferromagnetic Ising model. The same data are illustrated on a log-log plot of  $\phi$  vs  $K_c - K$  at the bottom.

by Onsager [32]. The results of the numerical simulations indicate similar order-disorder transitions for  $\epsilon = 0.2$  too. The similarity becomes more striking on the bottom log-log plot used frequently to demonstrate the power law behaviors. For the latter case, the order parameters are considered as a function of  $[K_c(\epsilon) - K]$ . A nice coincidence emerges if we choose  $K_c(\epsilon = 0.2) = 0.9274(1)$ . Similar universal critical behavior occurs in the antiferromagnetic Ising model in the presence of a homogeneous external magnetic field [33,34]. This universal behavior is justified for many other two-dimensional equilibrium systems. At the same time, the appearance of this behavior is not evident in the present nonequilibrium system where the detailed balance is broken as demonstrated in Fig. 6. According to the MC results, the specific entropy production refers to a relevant deviation from the detailed balance particularly within a wide region around the critical transition point. Notice, furthermore, the difference between the two approaches that refers to the relevance of the second neighbors in the elementary strategy reversals that is operating via some memory effects in the spatiotemporal patterns.

For low levels of noise, the formation of the sublattice ordered strategy arrangement reduces the frequency of strategy

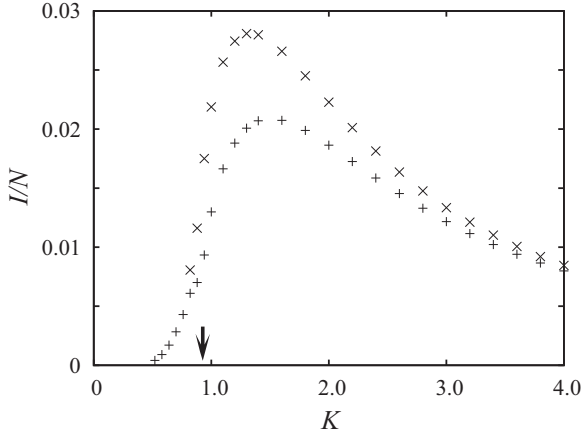


FIG. 6. Estimations for specific entropy production vs  $K$  on the square lattice for  $\epsilon = 0.2$  when only the first (+) and the first and second neighbor configurations ( $\times$ ) are taken into account during the counting of elementary events. The arrow indicates the transition point at  $K_c(\epsilon = 0.2)$ .

reversals as quantified in Figs. 5 and 6. The visualization of the time-dependent strategy distributions indicates that, in the ordered structure, isolated players rarely adopt undesired strategy and reverse back to the favored state within a short time. This phenomenon weakly reduces the order parameter and does not contribute to the specific entropy production. The frequency and size of these defects increase with the magnitude of noise  $K$ , and the sublattice ordered structure is completely destroyed above the critical point  $K_c(\epsilon)$ , whereas, the emerging specific entropy production is suppressed by the dominance of randomness at high noise levels.

Using MC simulations, we have evaluated the order parameter  $\phi$  for this model on bipartite random regular graphs when varying the noise level. The results are shown in Fig. 7 where  $\phi^2$  is plotted as a function of  $K$ . One can recognize that  $\phi^2$  vanishes linearly at the critical transition points for both  $\epsilon = 0$  and  $\epsilon = 0.2$ . These features refer to a mean-field type behavior

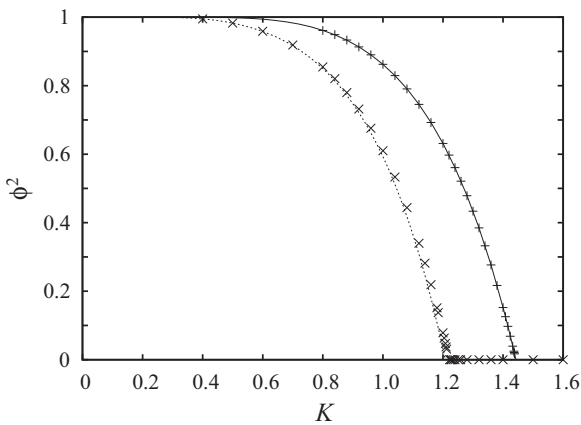


FIG. 7. Square of order parameter vs  $K$  for the anticoordination game on a bipartite random regular graph. Symbols + and  $\times$  show MC data for  $\epsilon = 0.0$  and  $\epsilon = 0.2$ , whereas, solid and dashed lines illustrate the prediction of two-site approximation for the same strength of the matching-pennies component.

( $\beta = 1/2$ ) that is predicted by theories for sufficiently high spatial dimensions and small-world structures [35]. Notice that the MC data (for  $\epsilon = 0$ ) reproduce the exact results provided by the two-site dynamical cluster method [10,35,36]. Additionally, this approach gives an adequate description for the disturbance of matching-pennies contribution. The small differences between the results of MC simulations and pair approximations refer to the emergence of additional long range correlations occurring for  $\epsilon = 0.2$ .

For the Ising type critical behaviors, there are other quantities (e.g., susceptibility, correlation length, and relaxation time) exhibiting power law divergence when approaching the critical transition point  $K_c$ . The corresponding exponents, however, are related to each other via the scaling laws as justified theoretically and by simulations [30]. At the same time, for the clear identification of the Ising universality class, one needs to determine another critical exponent too. This is the reason why we have evaluated the power law divergency of the order parameter fluctuations defined as

$$\chi(K) = N \langle [m(t) - \langle m(t) \rangle_t]^2 \rangle_t, \quad (12)$$

where  $m(t)$  is given by (9). This quantity becomes independent of  $N$  if  $N \rightarrow \infty$  and diverges as

$$\chi(K) \propto |K - K_c|^{-\gamma}, \quad (13)$$

if  $K$  approaches  $K_c$  from both sides of the critical point. The values of the exponent  $\gamma$  are known for the Ising model on both the square and the Bethe lattices, namely,  $\gamma = 7/4$  and 1, respectively. The MC results confirm the same universal behavior within the statistical error (comparable with the symbol size for the plotted data) in the case of  $\epsilon = 0.2$  with  $K_c(\epsilon = 0.2) = 1.2185(2)$  as demonstrated in Fig. 8. Similar results are found for  $\epsilon = 0$  when  $K_c(\epsilon = 0) = 1.44272(1)$ . The justification for this universal behavior for smaller values of  $|K - K_c|$  is prevented by the technical difficulties due to the diverging correlation length and time if  $K \rightarrow K_c(\epsilon)$ .

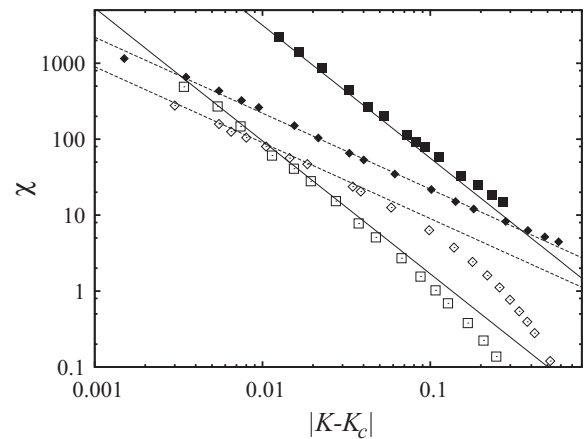


FIG. 8. Log-log plot of the order parameter fluctuation  $\chi(K)$  vs  $|K - K_c(\epsilon)|$  on the square lattice (boxes) and bipartite random regular graph (diamonds) for  $\epsilon = 0.2$ . The closed and open symbols denote MC data collected below and above the critical points. The solid and dashed lines (with two different slopes) illustrate the theoretical predictions for the asymptotic behaviors.

## VI. SUMMARY

We have studied multiagent evolutionary games on bipartite regular networks with a Glauber type dynamical rule when the effect of the anticonoordination type interactions is disturbed by the presence of matching-pennies games. First it is shown that the matching-pennies game itself drives this system into random strategy distribution decorated with weak correlations between the second and the third neighbors on the spatial system. These weak correlations depend on the topological features of the network and can be qualitatively well described by the dynamical cluster methods if we use sufficiently large clusters.

The appearance of these correlations may be related to the breaking of detailed balance producing probability currents between most of the forward-backward transitions between suitable pairs of microscopic states. These probability currents satisfy the Kirchhoff law in the stationary state and distort the probability distribution of states. As a result, the emerging correlations depend on the topology of network the game is investigated on. Now we have compared only two regular networks of the same degree.

In light of the above results, the matching-pennies component for the given dynamics can be considered as a microscopic driving force creating entangled probability current loops

within the microscopic states. This phenomenon becomes more striking when studying its effect on the ordering governed by an anticonoordination game. Our numerical and analytical methods support that a weak matching-pennies component does not destroy the formation of the sublattice ordered strategy arrangement at low noise levels. If the noise is increased, then this system exhibits an order-disorder transition at a critical point decreasing with the strength of the matching-pennies component. Despite the breaking of detailed balance, these transitions seem to belong to the Ising universality class on both the square lattice and the bipartite random regular graph.

Finally, we briefly mention that the preliminary investigations indicate further interesting and unexpected results when studying the effect of the matching-pennies game on the ordering phenomena for other types of games.

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- [1] M. A. Nowak, *Evolutionary Dynamics* (Harvard University Press, Cambridge, MA, 2006).
  - [2] G. Szabó and G. Fáth, *Phys. Rep.* **446**, 97 (2007).
  - [3] J. von Neumann and O. Morgenstern, *Theory of Games and Economic Behaviour* (Princeton University Press, Princeton, 1944).
  - [4] L. van Valen, *Evolutionary Theory* **1**, 1 (1973).
  - [5] L. van Valen, *Evolutionary Theory* **4**, 129 (1980).
  - [6] J. Maynard Smith and G. R. Price, *Nature (London)* **246**, 15 (1973).
  - [7] D. Friedman, *Econometrica* **59**, 637 (1991).
  - [8] R. Cressman, W. G. Morrison, and J.-F. Wen, *Can. J. Econ.* **31**, 1101 (1998).
  - [9] E. Ising, *Z. Phys.* **31**, 253 (1925).
  - [10] C. Domb, in *Phase Transitions and Critical Phenomena*, edited by C. Domb and M. S. Green (Academic, London, 1974), Vol. 3, pp. 357–484.
  - [11] S. Galam, Y. Gefen, and Y. Shapir, *Math. J. Sociol.* **9**, 1 (1982).
  - [12] G. Weisbuch and D. Stauffer, *Physica A* **384**, 542 (2007).
  - [13] S. Grauwin, D. Hunt, E. Bertin, and P. Jensen, *Adv. Complex Syst.* **14**, 529 (2011).
  - [14] S. M. Krause and S. Bornholdt, *Physica A* **392**, 4048 (2013).
  - [15] A. V. M. Herz, *J. Theor. Biol.* **169**, 65 (1994).
  - [16] R. J. Glauber, *J. Math. Phys.* **4**, 294 (1963).
  - [17] L. E. Blume, *Games Econ. Behav.* **5**, 387 (1993).
  - [18] L. E. Blume, *Games Econ. Behav.* **11**, 111 (1995).
  - [19] D. Monderer and L. S. Shapley, *Games Econ. Behav.* **14**, 124 (1996).
  - [20] D. Monderer and L. S. Shapley, *J. Econ. Theory* **68**, 258 (1996).
  - [21] H. A. Gutowitz, J. D. Victor, and B. W. Knight, *Physica D* **28**, 18 (1987).
  - [22] R. Dickman, *Phys. Rev. A* **38**, 2588 (1988).
  - [23] G. Szabó, J. Vukov, and A. Szolnoki, *Phys. Rev. E* **72**, 047107 (2005).
  - [24] R. Dickman, *Phys. Rev. E* **66**, 036122 (2002).
  - [25] B. Bollobás, *Random Graphs* (Academic, New York, 1985).
  - [26] J. Vukov, G. Szabó, and A. Szolnoki, *Phys. Rev. E* **73**, 067103 (2006).
  - [27] J. Schnakenberg, *Rev. Mod. Phys.* **48**, 571 (1976).
  - [28] G. Kirchhoff, *Poggendorff's Ann. Phys. Chem.* **72**, 495 (1847).
  - [29] G. Szabó, T. Tomé, and I. Borsos, *Phys. Rev. E* **82**, 011105 (2010).
  - [30] H. E. Stanley, *Introduction to Phase Transitions and Critical Phenomena* (Clarendon, Oxford, 1971).
  - [31] K. Toda, R. Kubo, and N. Saito, *Statistical Physics I: Equilibrium Statistical Mechanics* (Springer, Berlin/Heidelberg, 1991).
  - [32] L. Onsager, *Phys. Rev.* **65**, 117 (1944).
  - [33] R. B. Griffiths, *Phys. Rev. Lett.* **24**, 1479 (1970).
  - [34] D. C. Rapaport and C. Domb, *J. Phys. C* **4**, 2684 (1971).
  - [35] S. N. Dorogovtsev, A. V. Goltsev, and J. F. F. Mendes, *Rev. Mod. Phys.* **80**, 1275 (2008).
  - [36] R. J. Baxter, *Exactly Solved Models in Statistical Mechanics* (Academic, London, 1998).