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# Games, graphs and Kirchhoff laws

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## HIGHLIGHTS

- Relationships between potential games and graphs are developed.
- The existence of potential is related to orthogonality criteria between the game and a subset of voluntary matching pennies games.
- It is shown that each voluntary rock–paper–scissors game can be built up from two suitable voluntary matching pennies games.

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## ABSTRACT

Evolutionary potential games represent a set of biological and ecological models equivalent to multiparticle physical systems for a suitable dynamical rule. In these systems the pair interaction is described by a payoff matrix of two-player games possessing a wider class of interactions. Potential games satisfy criteria related to the Kirchhoff laws and have pure Nash equilibria. Using the bi-matrix formalism of game theory we show a simple method for checking the existence of potential which is related to the absence of cyclic components. It will be shown that potential exists if the game is orthogonal to a suitable set of cycling elementary games resembling voluntary matching pennies games. Relationships among these cyclic components and consequences of player's equivalence are also discussed.


## 1. Introduction

In spatial evolutionary games [1–4] players are located at the sites of a lattice and their income comes from two-player games with their neighbors. In these systems the players can represent biological objects, social agents, and particles when studying ecological, economic, and physical systems. The pair interactions between the equivalent neighbors are described by a payoff matrix with elements quantifying the incomes received by them when choosing one of their $n$ options, henceforth called strategies [5,6]. These strategies identify the types of species on each site of the biological system, the selected human behavior (e.g., selfish or altruistic) in social models, and the microscopic states of atoms forming crystalline structure. The payoffs measure the capability of creating offsprings (fitness) [7], the strength of motivation to choose one of the strategies [8], and the negative potential energy in the systems mentioned. The similar mathematical background has motivated the application of the concepts and methods of the statistical physics for the systematic investigation of many complex systems in the last decades.

We have to emphasize that the direct analogy between evolutionary games and thermodynamical systems is ensured if the pair interactions are described by the so-called potential games [9–11] and the evolution of strategy distribution is controlled by a logit rule [8,12] favoring the choice of strategies with a probability increasing exponentially with the

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individual income for the consecutive unilateral strategy updates. In fact, the latter systems evolve into a Boltzmann distribution [10,13–16]. Additionally, the results of mean-field approximations can well describe the system behavior if we ignore the underlying connectivity structure.

Most of the previous analyses are focused on systems where the pair interactions are defined by symmetric one-shot non-cooperative games [5,6,17]. For these games the players are equivalent and they exchange payoffs when exchanging strategies. Their possible incomes can be defined by a single payoff matrix $\mathbf{A}$. If this payoff matrix is symmetric ($\mathbf{A} = \mathbf{A}^T$) then the players share the payoffs fraternally eliminating the difference between the individual and community interest. Furthermore, it is a potential game with a potential matrix $\mathbf{V} = \mathbf{A}$.

In the knowledge of the potential matrix one can determine the preferred Nash equilibrium (defined by the largest matrix element of $\mathbf{V}$) playing the role of ground state in the corresponding physical system. For non-symmetric payoff matrices the existence of potential, as well as the evaluation of the potential, were clarified by exploiting the concept of matrix decomposition [18–22]. Similarly to basis vectors, this approach builds up the matrix from a suitable set of basis matrices classified into four orthogonal types of interactions: namely, games with self- and cross-dependent payoffs, coordinations, and cyclic components. In this approach the $n \times n$ matrices are considered as $n^2$-dimensional vectors that implies the introduction scalar product, orthogonality, and Cartesian coordinate systems. The four orthogonal classes of pair interactions become visible for a rotated coordinate system with axis representing elementary games of types mentioned above. The presence of any cyclic components prevents the existence of potential and also the thermodynamical behavior for the application of logit rule.

In other words, for symmetric $n$-strategy potential games the payoff matrix is orthogonal to all the independent cyclic elementary games involving a rock–paper–scissors type three-strategy subgame and $(n-3)$ neutral strategies. These elementary games can be considered as voluntary rock–paper–scissors game where the choice of the neutral strategies provide zero income for both players. An independent set of elementary cyclic components can be selected by those $(n-1)(n-2)/2$ voluntary rock–paper–scissors components which includes the first strategy. Evidently, one can choose other independent subsets of rock–paper–scissors subgame components when the distinguished role of the first strategy is replaced by another one.

Now we extend the above approach by considering the so-called bi-matrix games where the incomes of the distinguishable players are defined by two payoff matrices: $\mathbf{A}$ and $\mathbf{B}$. Using a pedagogical style it will be shown that for these more general systems the role of the voluntary rock–paper–scissors components can be replaced by simpler cyclic components representing voluntary matching-pennies games where the two competing strategies are extended by additional neutral strategies.

## 2. Basic concepts of bi-matrix games

We study non-symmetric games with two players ($x$ and $y$) having $n$ and $m$ strategies. The players' incomes can be expressed by using the bi-matrix formalism [5,23–25], namely,

$$\mathbf{G}(\mathbf{A}, \mathbf{B}) = \begin{pmatrix} (A_{11}, B_{11}) & \cdots & (A_{1m}, B_{1m}) \\ \vdots & \ddots & \vdots \\ (A_{n1}, B_{n1}) & \cdots & (A_{nm}, B_{nm}) \end{pmatrix}. \tag{1}$$

In this notation the pair of bi-matrix components $(A_{ij}, B_{ij})$ refer to payoffs received by players $x$ and $y$ if they choose the strategy pair $(i, j)$ $(1 \le i \le n$ and $1 \le j \le m)$. In these games the (selfish and intelligent) players wish to maximize their own payoffs and are not allowed to communicate before choosing one of their strategies.

These games have one or more Nash equilibria [26] which are recommended for the players to choose because the unilateral deviations from these strategy pairs are not advantageous for the deviant player. In short, both players are satisfied under the given conditions. For a pure Nash equilibrium $(i^\star, j^\star)$ it means that

$$A_{i^\star j^\star} \ge A_{kj^\star} \text{ and } B_{i^\star j^\star} \ge B_{i^\star l} \tag{2}$$

for all possible values of $k$ and $l$ $(k = 1, \ldots, n$ and $l = 1, \ldots, m)$. For strict Nash equilibria the payoff of the unilateral deviant is decreased. Additionally, these systems can have one or more mixed Nash equilibria when the players can select one of the suitable strategies with different probabilities [5].

A simple and attractive way of finding the possible pure Nash equilibria is based on the derivation of flow graph. In flow graphs [27], and also in dynamical graphs [28], the nodes denote strategy pairs arranged in a rectangular form in the same way as they occur in the bi-matrix (1). For both graphs the edges connect those strategy pairs where only one of the players has modified her strategy. The corresponding dynamical graphs characterize systems where consecutive unilateral strategy changes are allowed. In flow graphs the edges are directed and point toward the strategy pair providing higher income for the player modifying her strategy.

Fig. 1 shows three different flow graphs. In the first (a) example the symmetric three-strategy game has three pure Nash equilibria. The second (b) example illustrates a three-strategy bi-matrix games where the payoffs are given by integers $(1, \ldots, 9)$ selected at random for both players. These examples illustrate, that for many real life situation it is enough to characterize the payoffs by integers reflecting the rank of preference for both players. The third (c) flow graph illustrates the preferred strategy changes the traditional rock–paper–scissors game.

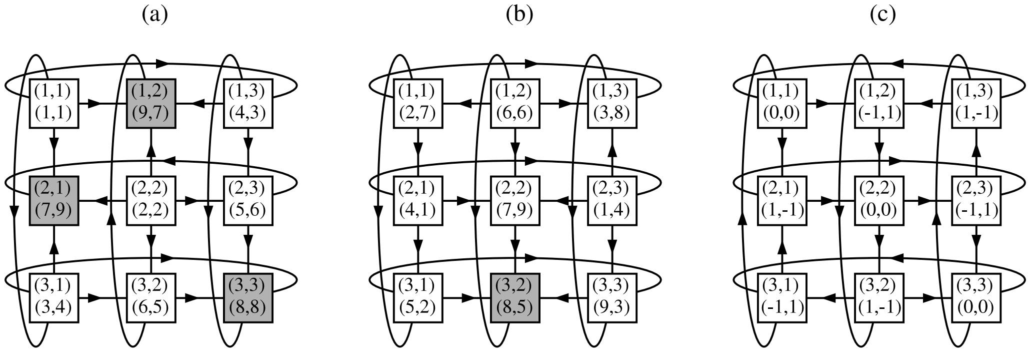


**Fig. 1.** Flow graphs for three-strategies in a symmetric game (a), in a bi-matrix game with payoffs chosen at random (b), and in the rock–paper–scissors game (c). The nodes are denoted by boxes in which the upper labels refer to strategy pairs, the lower figures indicate payoffs received by the players. The strict Nash equilibria are distinguished by gray color.

In a flow graph the strict Nash equilibria can be easily recognized because the corresponding nodes have only incoming directed edges. We emphasize, furthermore, that only one strict Nash equilibrium can exist in each row and column. This fact maximizes the number of pure and strict Nash equilibria [29–32]. In the absence of pure Nash equilibrium for the rock–paper–scissors game the Nash theorem [26,33] prescribes the existence of a mixed Nash equilibrium for which the players choose one of their three strategies with the same probabilities (1/3).

The directed graphs (a) and (b) (in Fig. 2) have no directed loops. In fact, the absence of directed loops ensures the existence of pure Nash equilibria which can be easily justified by using a simple graph theoretical argumentation [27,34–36]. Namely, if a directed path is started from a node of a directed graph by choosing one of the outgoing edges step by step then we cannot return to the nodes of the resulting path in the absence of directed loops. These directed paths end in nodes without outgoing edges. Otherwise, after $N-1$ steps ($N = nm$ is the number of nodes) we cannot find a unvisited node. If the directed path is defined by selecting one of the incoming edges consecutively then the same argumentation justify the existence of node(s) having only outgoing edges.

On the contrary, for the rock–paper–scissors game one can easily find four-node directed loops, for example, $(1,2) \to (2,2) \to (2,3) \to (1,3) \to (1,2)$.

The flow graph of a potential game is free of directed loops. If $\mathbf{G}$ is a potential game then we can derive a potential matrix

$$\mathbf{V}(\mathbf{A},\mathbf{B}) = \begin{pmatrix} V_{11} & \cdots & V_{1m} \\ \vdots & \ddots & \vdots \\ V_{n1} & \cdots & V_{nm} \end{pmatrix}. \tag{3}$$

with components satisfying the conditions:

$$\begin{aligned} V_{kj} - V_{ij} &= A_{kj} - A_{ij}\,, \\ V_{il} - V_{ij} &= B_{il} - B_{ij}\,, \end{aligned} \tag{4}$$

where $i, k = 1, \ldots, n$ and $j, l = 1, \ldots, m$. Notice, that a large portion of payoff parameters are dropped when deriving the potential. If potential exists then the elements $V_{ij}$ of the potential matrix can be evaluated by summing the payoff variations (given by Eqs. (4)) of the strategy-modifying player through unilateral strategy changes along a path through the strategy pairs of the dynamical graph. This approach is similar to those suggested by Miekisz [15].

The linear relationships (4) between the potential and payoff values can be exploited to explore some general features. For example, if a game is composed of two potential games, e.g., $\mathbf{G}(\mathbf{A},\mathbf{B})$ and $\mathbf{G}'(\mathbf{A}',\mathbf{B}')$ with payoff matrices $\mathbf{A}+\mathbf{A}'$ and $\mathbf{B}+\mathbf{B}'$ then the resultant game has a potential matrix given as $\mathbf{V}(\mathbf{A},\mathbf{B})+\mathbf{V}'(\mathbf{A}',\mathbf{B}')$. For a detailed analysis of the consequences of this feature we suggest consulting the papers [10,16]. Now we only emphasize that this approach utilizes the similarity between games (1) and vectors with a dimension of $2nm$. Accordingly, in addition to the sum of two games, one can introduce the scalar product of two games as

$$\mathbf{G}(\mathbf{A},\mathbf{B}) \cdot \mathbf{G}'(\mathbf{A}',\mathbf{B}') = \sum_{i,j} [A_{ij}A'_{ij} + B_{ij}B'_{ij}], \tag{5}$$

and the concept of orthogonality, as well. More precisely, the games $\mathbf{G}(\mathbf{A},\mathbf{B})$ and $\mathbf{G}'(\mathbf{A}',\mathbf{B}')$ are orthogonal to each other if

$$\mathbf{G}(\mathbf{A},\mathbf{B}) \cdot \mathbf{G}'(\mathbf{A}',\mathbf{B}') = 0. \tag{6}$$

The concept of matrix decomposition and the classification of symmetric games into four types of elementary (orthogonal) interactions [20] is based on the concepts mentioned. One can think that these elementary basis games serve as new basis vectors which span the whole space of payoff parameters and reflects the inherent symmetries. This approach can help us to study separately the effects of these fundamentally different pair interactions.

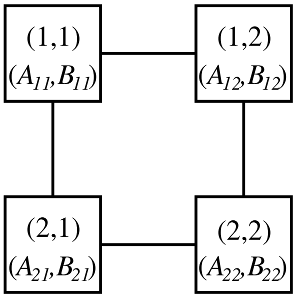


**Fig. 2.** The dynamical graph of a two-strategy game has four strategy pairs and four edges forming a single loop. The nodes are denoted with boxes including the same parameters as in Fig. 1.

## 3. Existence of potential for two-strategy games

The potential matrix summarizes the individual incentives for consecutive unilateral strategy changes. At the same time, the existence of potential requires that the potential variation $V_{i'j'} - V_{ij}$ should be independent of the paths (in the dynamical graph) defining how the players can get from the strategy pair $(i, j)$ to $(i', j')$ via unilateral consecutive steps. The latter statement is satisfied if the sum of the payoff variations of the active players is zero along all the possible closed loops of the dynamical graph.

For two-strategy games ($n = m = 2$) the dynamical graph (see Fig. 2) has only one loop that simplifies the problem, because we have only one condition to be satisfied for the existence of potential. Quantitatively, if we start from the strategy pair $(1, 1)$ and go clockwise then the sum of the payoff variations of the active players along this loops vanishes, if

$$B_{12} - B_{11} + A_{22} - A_{12} + B_{21} - B_{22} + A_{11} - A_{21} = 0. \tag{7}$$

If this single condition is satisfied by the payoff parameters ($A_{ij}$ and $B_{ij}$) then one can easily evaluate the elements of the potential matrix $\mathbf{V}$. Disregarding an irrelevant constant this quantity can be given as

$$\mathbf{V}(\mathbf{A}, \mathbf{B}) = \begin{pmatrix} A_{11} & (A_{11} + B_{12} - B_{11}) \\ A_{21} & (A_{21} + B_{22} - B_{21}) \end{pmatrix}. \tag{8}$$

Notice, that the first columns of the matrices $\mathbf{A}$ and $\mathbf{V}$ are identical and the missing two elements of $\mathbf{V}$ are evaluated by using the Eqs. (4). Evidently, the same method can be adapted for arbitrary values of $n$ and $m$ if the existence of potential is already justified.

For two-strategy games the potential $\mathbf{V}$ has only three relevant parameters because adding an irrelevant constant to each element ($V_{ij} \to V_{ij} + c$ ) is permitted by Eqs. (4). Similarly, some variations in the payoff matrices (e.g., $A_{ij} \to A_{ij} + a_j$ and $B_{ij} \to B_{ij} + b_i$ with arbitrary constants $a_j$ and $b_i$) will preserve the validity of Eq. (7) and the potential (8) remains unchanged. The latter additional terms can be considered a linear combination of elementary games, called games with cross-dependent payoffs [16,20], which give zero contribution to $\mathbf{V}$.

In a previous review [16] the reader can find a possible decomposition of the two-strategy bi-matrix games into the linear combinations of eight orthogonal elementary games. In this notation $\mathbf{G}(\mathbf{A}, \mathbf{B})$ is described as:

$$\mathbf{G}(\mathbf{A}, \mathbf{B}) = \sum_{p=1}^{8} \alpha(p) \mathbf{g}^{(\mathrm{p})}, \tag{9}$$

where the elementary games $\mathbf{g}^{(\mathrm{p})}$ are orthogonal to each other, that is, $\mathbf{g}^{(\mathrm{p})} \cdot \mathbf{g}^{(\mathrm{q})} = 0$, if $p \neq q$. For this decomposition there is only one elementary game (called matching pennies) which prevents the existence of the potential. The matching pennies game is a zero-sum game given as

$$\mathbf{g}^{(\mathrm{mp})} = \begin{pmatrix} (1, -1) & (-1, 1) \\ (-1, 1) & (1, -1) \end{pmatrix}. \tag{10}$$

Using this expression, a two-strategy bi-matrix game (given by (1) for $n = m = 2$) is a potential games if it is orthogonal to $\mathbf{g}^{(\mathrm{mp})}$, that is, $\mathbf{G}(\mathbf{A}, \mathbf{B}) \cdot \mathbf{g}^{(\mathrm{mp})} = 0$. This criterium coincides with the condition (7) derived above.

For the game defined by $\mathbf{g}^{(\mathrm{mp})}$ one of the players is always forced to reverse his/her strategy unilaterally that yields a clockwise circulation along the single four-edge loop (see Fig. 2) if they are allowed to change strategy alternately (or in random order). The effect of this term can be interpreted as a driving force of circulation which destroys the detailed balance if a stochastic logit rule is applied to games on networks. Additionally, this update creates entropy production [19,28]. Anyway, the matching pennies game has a single mixed Nash equilibrium where the players choose one of the strategy at random and independent of each other.

In the subsequent section we show that similar features and relationships are inherited for the multistrategy games.

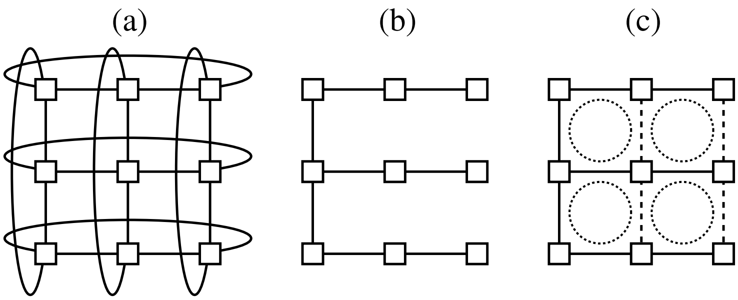


**Fig. 3.** The dynamical graph (a) and a possible spanning tree (b) for $n = m = 3$. In the graph (c) the dashed lines indicate four additional edges while the dotted circles illustrate the simplest four loops selected. Boxes without labels refer to strategy pairs in the same way as before.

## 4. Multistrategy bi-matrix games

When extending the previous analysis for the multistrategy games ($n, m > 2$) then we face a problem related to the large number of loops in the dynamical graph (see plot (a) in Fig. 1) which are not independent. The emerging difficulties, however, are reduced by the symmetries occurring in the dynamical graphs, too. Notice, that each subgraph is complete if it contains the nodes of a row (or column) with the suitable edges. Furthermore, this structure is resembling a square lattice with periodic boundary conditions and with additional edges between any horizontal or vertical pairs of nodes. These facts are consistent with the symmetries ensuring that the translation (or permutation) of strategy labels does not change the possible behaviors in this set of games.

The Kirchhoff laws [37] serve as a theoretical background to overcome these difficulties. Similar difficulties emerge for the quantitative analysis of electronic circuits. Now we adopt and apply the standard recipe developed to determine the number of independent loops [27,38] and select suitable ones. Accordingly, the number of independent loops is equivalent to the number of deleted edges when we reduce the graph to a spanning tree. Elementary calculation gives that the number of edges of the dynamical graph is nm$(n-1)(m-1)/2$ while the number of edges of the spanning tree is $nm-1$.

Fig. 3 compares the dynamical graph and a possible spanning tree for $n = m = 3$. The main steps of the sketched algorithm can trivially be extended to the cases with $n, m > 3$. The independent loops can be selected via the shortest loop including one of the missing edges we add when extending the spanning tree step by step. Fig. 3 illustrates how the spanning tree is extended by four edges and four independent loops, if $n = m = 3$. In general, for $n, m > 3$, the same algorithm extends the spanning tree with $(n-1)(m-1)$ new edges and four-edge loops. In fact, these loops define the relevant and independent set of loops because any other new edges belong to a loop containing nodes within a single row or column. For the latter cases only one player changes strategy and the sum of his/her payoff variations is zero for each (vertical or horizontal) loop.

For the existence of potential the upper-left four-edge loop in Fig. 3(c) defines a criterium equivalent to those given by Eq. (7), and similar expressions can be derived for the other relevant and independent four-edge loops. These criteria, however, can also be expressed via the help of orthogonality, as before. For this goal we introduce a set of voluntary matching pennies games defined by bi-matrices (8) which are extended with $(n-2)$ rows and $(m-2)$ columns containing 0s. Quantitatively, for the voluntary matching pennies games $\mathbf{g}^{(\mathrm{vmp})}((i,j),(i',j')) = \mathbf{g}^{(\mathrm{vmp})}((i',j'),(i,j))$ and player $x$ wins 1 from $y$ for the strategy profiles $(i,j)$ and $(i',j')$; the chances are equalized by the strategy pairs $(i,j')$ and $(i',j)$ when player $y$ gets 1 from $x$. For this definition we assumed that $i < i'$ and $j < j'$. If the order of strategy labels ($i \leftrightarrow i'$ or $j \leftrightarrow j'$) is reversed for one of the players then their payoffs are exchanged, quantitatively, $\mathbf{g}^{(\mathrm{vmp})}((i,j'),(i',j)) = \mathbf{g}^{(\mathrm{vmp})}((i',j),(i,j')) = -\mathbf{g}^{(\mathrm{vmp})}((i,j),(i',j'))$. For these elementary games the players $x$ and $y$ have $(n-2)$ and $(m-2)$ additional options to decline participation. They receive nothing if (at least) one of them chooses a neutral strategy. To keep the formulas simple, here the parameters $n$ and $m$ are omitted.

Using this set of elementary games, the existence of the potential can also be expressed by $(n-1)(m-1)$ orthogonality relations:

$$\mathbf{G}(\mathbf{A},\mathbf{B}) \cdot \mathbf{g}^{(\mathrm{vmp})}((i,j),(i+1,j+1)) = 0, \tag{11}$$

where $1 \le i < n$ and $1 \le j < m$. We emphasize, that the above orthogonality conditions are restricted to an independent and complete subset of the voluntary matching pennies games. At the same time it means, that there are many other voluntary matching-pennies games which can be built up from those used in (11). For example,

$$\mathbf{g}^{(\mathrm{vmp})}((i,j),(i+p,j+q)) = \sum_{p'=i}^{i+p-1} \sum_{q'=j}^{j+q-1} \mathbf{g}^{(\mathrm{vmp})}((p',q'),(p'+1,q'+1)), \tag{12}$$

where $p, q > 1$, while $i + p < n, j + q < m$.

For the independent and complete subset of voluntary matching pennies games defined by Eqs. (11) the strategy pair $(1, 1)$ plays a distinguished role which can be replaced by any other one (e.g., by $(i, j)$) for a suitable cyclic permutation

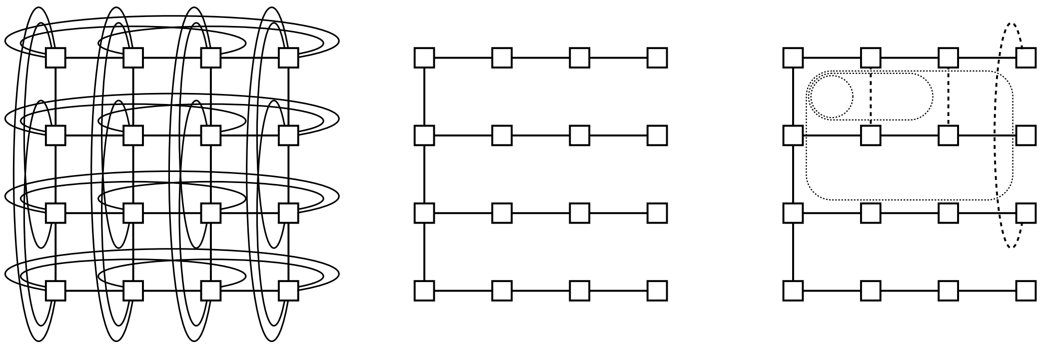


**Fig. 4.** The dynamical graph (a) and a spanning tree (b) for $n = m = 4$. Plot (c) illustrates that adding three vertical edges (illustrated with dashed lines) can generate three four-edge loops (dotted lines) including strategy pair (1,1). To avoid confusion the horizontal and vertical loops (related to trivial conditions) are not shown.

of the labels. By this way one can introduce further independent and complete subsets of voluntary matching pennies games. Evidently, the permutation of the strategy labels yields further variants. The common features of these subsets of elementary components is that here the players play voluntary matching pennies games with neighboring strategies ($\mathbf{g}^{(\mathrm{vmp})}((i,j),(i+1,j+1))$).

Following the same recipe one can easily define another subset of voluntary matching pennies games which is consistent with the orthogonality criteria derived previously for the symmetric games [21]. Fig. 4 summarizes the essence of another algorithm for $n = m = 4$. Now, the spanning tree is similar to those used in Fig. 3 and it can be extended with all the missing horizontal edges in each row without receiving nontrivial conditions. Similarly, the addition of all the missing vertical edges in the first column yields only trivial conditions. To avoid confusion the additions of these edges are not denoted in Fig. 4. The additions of the relevant edges (denoted by dashed line in Fig. 4(c)) connect the strategy pairs $(1, j)$ and $(i, j)\,(1 < i \le n$ and $1 < j \le m)$ giving $(n-1)(m-1)$ conditions for the existence of potential along the four-edge loops $(1, 1) \to (1, j) \to (i, j) \to (i, 1) \to (1, 1)$. These conditions are equivalent to the following orthogonality conditions:

$$\mathbf{G}(\mathbf{A},\mathbf{B}) \cdot \mathbf{g}^{(\mathrm{vmp})}((1,1),(i,j)) = 0 \tag{13}$$

for the $i$ and $j$ values given above.

In the light of the above results a bi-matrix game can also be separated into the sum of a potential game $\mathbf{G}^{(\mathrm{pot})}$ and a cyclic component $\mathbf{G}^{(\mathrm{cyc})}$ ($\mathbf{G}(\mathbf{A},\mathbf{B}) = \mathbf{G}^{(\mathrm{pot})} + \mathbf{G}^{(\mathrm{cyc})}$) in a way that $\mathbf{G}^{(\mathrm{pot})} \cdot \mathbf{G}^{(\mathrm{cyc})} = 0$ where the cyclic component is the linear combination of a suitable subset of voluntary matching pennies games. For example,

$$\mathbf{G}^{(\mathrm{cyc})} = \sum_{i=2}^{n}\sum_{j=2}^{m} \alpha(i,j)\mathbf{g}^{(\mathrm{vmp})}((1,1),(i,j)). \tag{14}$$

In the next section we discuss the consequences of the above features for the symmetric matrix games where the cyclic component can be built up as the linear combination of a suitable set of voluntary rock–paper–scissors games. The comparison forecasts an intimate relationship between the voluntary matching pennies and rock–paper–scissors games.

## 5. Symmetric multistrategy games

For symmetric games $\mathbf{B} = \mathbf{A}^T$ and $n = m$, thus the payoffs can be defined by a single payoff matrix $\mathbf{A}$. In these games the choice of the strategy pair $(i, j)$ provides payoffs $A_{ij}$ and $A_{ji}$ for the first and second players, respectively.

First we emphasize that the criterium (3) is always satisfied for the symmetric two-strategy ($n = m = 2$) games. This is the reason why the symmetric two-strategy games are potential games.

According to Eqs. (14), for the symmetric three-strategy ($n = m = 3$) games potential exists if four criteria are satisfied. On the contrary, the previous analyses [16,20,21] predicted only one criterium to be satisfied. The apparent contradiction can be resolved by recognizing that $\mathbf{G}(\mathbf{A},\mathbf{A}^T) \cdot \mathbf{g}^{(\mathrm{vmp})}((1,1),(i,i)) = 0$ for each payoff matrix $\mathbf{A}$. In fact, the latter criterium is a generalized version of the previous one and it is valid for all the symmetric two-strategy subgames when both players are constrained to use the same two strategies, here 1 or $i$. This feature is related to the fact that for symmetric games the players with equivalent strategy (e.g., $(i, i)$) receive equivalent payoffs ($A_{ii}$), and the unilateral deviation from this strategy pair yields a payoff increase $A_{ji} - A_{ii}$ for any one choosing strategy $j$. The latter feature ensures that two of the four criterium (14) are satisfied due to the symmetries mentioned. On the other hand, the remaining two criteria coincide and can be replaced by single one, namely

$$A_{12} - A_{21} + A_{23} - A_{32} + A_{31} - A_{13} = 0, \tag{15}$$

which expresses the orthogonality ($\mathbf{A} \cdot \mathbf{A}^{(\mathrm{rsp})} = 0$) between $\mathbf{A}$ and the payoff matrix $\mathbf{A}^{(\mathrm{rsp})}$ of the traditional rock–paper–scissors game.

In the light of the above results it is not surprising that the rock–paper–scissors game can be built up from two three-strategy voluntary matching pennies games, that is, $\mathbf{g}^{(\mathrm{rps})} = \mathbf{g}^{(\mathrm{vmp})}((1,2),(2,3)) - \mathbf{g}^{(\mathrm{vmp})}((2,1),(3,2))$. The validity of this relation can be easily checked with using the bi-matrix formalism, that is,

$$\mathbf{g}^{(\mathrm{rps})} = \begin{pmatrix} (0,0) & (1,-1) & (-1,1) \\ (-1,1) & (0,0) & (1,-1) \\ (1,-1) & (-1,1) & (0,0) \end{pmatrix} = \begin{pmatrix} (0,0) & (1,-1) & (-1,1) \\ (0,0) & (-1,1) & (1,-1) \\ (0,0) & (0,0) & (0,0) \end{pmatrix} - \begin{pmatrix} (0,0) & (0,0) & (0,0) \\ (1,-1) & (-1,1) & (0,0) \\ (-1,1) & (1,-1) & (0,0) \end{pmatrix}. \tag{16}$$

Evidently, two other decompositions can be obtained by cyclic permutation of labels. For example, $\mathbf{g}^{(\mathrm{rps})} = \mathbf{g}^{(\mathrm{vmp})}((2,3),(3,1)) - \mathbf{g}^{(\mathrm{vmp})}((3,2),(1,3))$.

For multistrategy ($n = m > 3$) symmetric games some of the above features are preserved in a generalized form. For example, $\mathbf{G}(\mathbf{A}, \mathbf{A}^T) \cdot \mathbf{g}^{(\mathrm{vmp})}((i,i),(j,j)) = 0\,(1 \le i < j \le n)$. The existence of the potential requires additional conditions which may coincide. More precisely, $\mathbf{G}(\mathbf{A}, \mathbf{A}^T) \cdot \mathbf{g}^{(\mathrm{vmp})}((i,j),(i',j')) = -\mathbf{G}(\mathbf{A}, \mathbf{A}^T) \cdot \mathbf{g}^{(\mathrm{vmp})}((j,i),(j',i')) = 0$ in the symmetric potential games for a matrix $\mathbf{A}$. In order to reduce number of criteria the latter relationships can be exploited if we introduce the voluntary rock–paper–scissors games. More precisely, the elementary game $\mathbf{g}^{(\mathrm{vrsp})}(i,j,k)$ denotes an $n$-strategy symmetric game involving a rock–paper–scissors type cyclic dominance among the strategies $i, j$, and $k$ ($1 \le i < j < k \le n$) and $(n-3)$ neutral strategies providing zero payoffs for both players as before. In other words, $\mathbf{g}^{(\mathrm{vrsp})}(i,j,k)$ can be obtained from $\mathbf{g}^{(\mathrm{rps})}$ (given by Eq. (16) ) by adding $(n-3)$ suitable rows and columns with pairs of zeros.

Using the voluntary rock–paper–scissors games a potential exists if the game satisfies the following orthogonality criteria:

$$\mathbf{G}(\mathbf{A}, \mathbf{A}^T) \cdot \mathbf{g}^{(\mathrm{vrsp})}(1,i,j) = 0 \tag{17}$$

for $1 \le i < j \le n$. Thus, due to symmetry the number of independent orthogonality criteria is reduced to $(n-1)(n-2)/2$ (from $(n-1)^2$) in agreement with previous results [21]. Additionally, the calculations can be simplified as detailed in the paper mentioned.

Finally we mention that each voluntary rock–paper–scissors game can also be built up from two voluntary matching pennies games, as it is illustrated by Eq. (16) for $n = 3$. Quantitatively,

$$\mathbf{g}^{(\mathrm{vrps})}(i,j,k) = \mathbf{g}^{(\mathrm{vmp})}((i,j),(j,k)) - \mathbf{g}^{(\mathrm{vmp})}((j,i),(k,j)) \tag{18}$$

if $1 \le i < j < k \le n$ and also for the cyclic permutation of indices $i, j$, and $k$.

## 6. Summary

We have studied the existence potential in bi-matrix games which requires the systematic analysis of the sum of payoff variation along the loops of the dynamical graph consisting of strategy pairs (nodes) and unilateral changes (edges). The present graph theoretical approach is based on the standard method developed for the investigation of electronic circuits. The technical difficulties of this problem is related to the large number of loops to be tested. At the same time the symmetries of the dynamical graph and also the peculiar features of the games have simplified the solution.

Similar analyses were performed previously for the symmetric $n$-strategy game. Now the analyses are extended for the investigation of bi-matrix games. It turned out that the use of the more general bi-matrix formalism simplifies the application of the method mentioned. Thus we have $(n-1)(m-1)$ independent conditions (equations) imposed by the existence of potential in the bi-matrix games if the two players have $n$ and $m$ strategies. It is found that the corresponding equations are equivalent to orthogonality criteria between the bi-matrix game and a suitable subset of voluntary matching pennies games we introduced.

The present results are contrasted with those obtained previously for the symmetric games where the existence of potential is related to simpler orthogonality conditions between the actual payoff matrix ($\mathbf{A}$) and a suitable subset of payoff matrices corresponding to the voluntary rock–paper–scissors games. According to the concept of matrix decomposition a symmetric game can be considered as a sum of a potential game and a cyclic game where the latter one is the linear combination of an independent subset of voluntary rock–paper–scissors games. In the light of the present results, for the bi-matrix games similar decomposition exists and here the cyclic components are constructed as a linear combinations of an independent subset of voluntary matching pennies games.

Symmetric games can be considered as a special case of bi-matrix games (when $\mathbf{B} = \mathbf{A}^T$) which implies relevant relationships between the voluntary rock–paper–scissors and matching pennies games. Now it is found that any voluntary rock–paper–scissors game can be built up from two suitable voluntary matching pennies games.

Finally we underline the relevance and usefulness of the graph theoretical background we used. For the illustration of the wide scale of its applicability we have briefly surveyed some results related to the analysis of the pure Nash equilibria existing for the potential games. It is demonstrated that the corresponding graphs represent clearly the inherent symmetries, the entanglement of loops, and distinguish the relevant and irrelevant (trivial) loops in the dynamical graphs. Now, these tools helped us to find different independent and complete sets of the voluntary matching pennies games.

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