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The role of mixed strategies in spatial evolutionary games

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ABSTRACT

We study three-strategy evolutionary games on a square lattice when the third strategy is a mixed strategy of the first and second ones. It is shown that the resultant three-strategy game is a potential game as well as its two-strategy version. Evaluating the potential we derive a phase diagram on a two-dimensional plane of rescaled payoff parameters that is valid in the zero noise limit of the logit dynamical rule. The mixed strategy is missing in this phase diagram. The effects of two different dynamical rules are analyzed by Monte Carlo simulations and the results of imitation dynamics indicate the dominance of the mixed strategy within the region of the hawk–dove game where it is an evolutionarily stable strategy. The effects and consequences of the different dynamical rules on the final stationary states and phase transitions are discussed.

1. Introduction

In most of the multi-agent evolutionary games [1–3] the interactions among the equivalent players are described by a suitable sum of symmetric two-person games characterized by a uniform $n \times n$ payoff matrix $\mathbf{A}$ [4] where n is the number of strategies. In the spatial evolutionary games the players are distributed on a lattice and their individual income comes from games played with their nearest neighbors. In these models the players are allowed to modify their own strategy by following a dynamical rule that may be based on the deterministic [5] or stochastic [6] imitation of a better (or the best) neighbor, or favoring the selection of a better strategy with assuming fixed strategies in the neighborhood [7–11], *etc.*. In the last years the models with a logit dynamical rule are investigated more and more frequently in the literature of physics (for a recent review see Ref. [12]) because this dynamical rule drives the systems into a Boltzmann distribution if the pair interactions are defined by potential games [13–16]. The similarity between the Ising type models [17,18] and some two-strategy evolutionary games was reported previously by many authors [19–24].

In normal games the players have a finite number of pure strategies and each player has a quantified payoff function dependent on her own strategy and also on the strategies chosen by the co-players. Evidently, these games involve different types of interactions [25,12] including situations when conflicts can occur between the interest of selfish (rational) players who wish to maximize their own payoff irrespective of the others [26,16]. For the potential games we can derive a potential, as a function of strategy profiles, that comprises the individual interest of the active player when only unilateral strategy changes are allowed in the consecutive steps during the evolutionary process.

The potential games have some curious features that make them attractive when drawing parallels between the evolutionary games and models of statistical physics. First we emphasize that the potential games have one or more pure

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Nash equilibria from which the unilateral deviation is not beneficial for the active player. If the potential has a single maximum then the corresponding strategy profile can be considered as a preferred pure Nash equilibrium to which the system evolves in the low noise limit for the logit rule. The concept of potential and preferred Nash equilibrium are analogous to the negative potential energy (Hamiltonian) and ground state of a many particle system.

Now we will study in parallel two- and three-strategy evolutionary games with players located on a square lattice and playing games with their nearest neighbors. For the definition of the symmetric two-strategy games we use the notation of social dilemmas with rescaled payoffs characterizing the interactions with only two parameters. All the latter evolutionary games are potential games which can be mapped onto a kinetic Ising model for the application of the logit rule. At the same time the two-strategy symmetric games have an additional mixed Nash equilibrium in certain regions of the two-dimensional parameter space. The mentioned mixed Nash equilibrium becomes relevant for the hawk–dove games where it is an evolutionarily stable strategy [27] and plays a crucial role in biological systems described by population dynamics [28]. Some additional features of the mixed strategies are considered in several previous papers [29–34]. Now the role of mixed strategies is analyzed by discussing a three-strategy game that is derived from the two-strategy games with introducing one of the mixed strategies as a third pure strategy. It is found that the resultant three-strategy game is also a potential game and the third (mixed) strategy plays a minor role in the evolutionary game at low noises if a logit rule controls the evolution of strategy distribution. On the contrary, the dominance of mixed strategies was indicated by the Monte Carlo simulations if the dynamics is governed by random sequential imitations of a better neighbor via stochastic pairwise comparison of payoffs.

In the next section we define the formalism and models, evaluate the potentials and phase diagrams in the zero noise limit, and discuss the general features of the mixed strategies. The numerical analysis of the spatial evolutionary games is focused on the determination of the strategy frequencies when varying the noise level for the logit rule and also on the evaluation of phase diagram for imitation at a low noise level. Additionally we will illustrate the appearance of consecutive phase transitions when one of the payoff parameters is tuned at a fixed (low) noise. The results are detailed in Section 3 by considering separately the consequences of logit rule and imitation. Finally we survey the main messages.

2. Formalism, models, and general features

2.1. Two-strategy games

In the present spatial evolutionary games the equivalent players are located at the sites x of a square lattice. For the two-strategy models each player chooses one of her two options, called pure strategies, denoted traditionally by two-dimensional unit vectors as

$$\mathbf{s}_x = D = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \text{or} \quad C = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{1}$$

where D and C refer to defection and cooperation for the prisoner's dilemma games. Using these strategies the players play games with their four nearest neighbors. The income (u_x and u_y) of two neighboring players (at sites x and y) are given by products,

$$u_x = \mathbf{s}_x \cdot \mathbf{A}\mathbf{s}_y \quad \text{and} \quad u_y = \mathbf{s}_y \cdot \mathbf{A}\mathbf{s}_x \tag{2}$$

where the element A_{ij} of the payoff matrix $\mathbf{A}$ defines the payoff for the first player if she chooses her ith strategy whereas the co-player selected the jth strategy ($i, j = 1, 2$ referring to D and C).

Using the notation of social dilemmas [26] with rescaled payoffs the payoff matrix $\mathbf{A}$ is described by two parameters as

$$\mathbf{A} = \begin{pmatrix} 0 & T \\ S & 1 \end{pmatrix} \tag{3}$$

where T defines the temptation to choose defection, S is the sucker's payoff, $R = 1$ is the reward for mutual cooperation, and $P = 0$ refers to punishment for mutual defections within the region of prisoner's dilemma where $T > 1$ $S < 0$. In the T–S plane of parameters four types of games are distinguished in the left plot of Fig. 1 by illustrating the corresponding flow graphs. In these flow graphs boxes with labels indicate strategy profiles and the directed edges connect strategy pairs where only one of the players modifies her strategy. The arrows point to the preferred strategy of the active player. Here the nodes without outgoing edges are Nash equilibria. This plot illustrates that in the regions of the prisoner's dilemma (PD) and Harmony (H) games there is only one pure Nash equilibrium and two pure Nash equilibria exist for the hawk–dove (HD) and stag hunt (SH) games. The boundaries ($S = 0$ and $T = 1$) between the different types of games are denoted by dashed–dotted lines.

The sum of the payoff variations is zero along the single four-edge loop for all the flow graphs of the symmetric two-player two-strategy games. Consequently, these games are potential games [13,15] and we can derive a symmetric potential matrix [12]

$$\mathbf{V} = \begin{pmatrix} 0 & S \\ S & 1 + S - T \end{pmatrix} \tag{4}$$

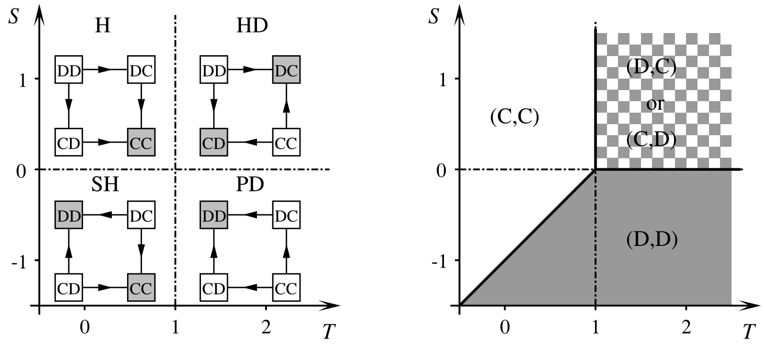

Fig. 1. Four flow graphs (left) characterize the games on the T–S plane. The horizontal ($S = 0$) and vertical ($T = 1$) dashed–dotted lines separate the regions of the harmony (H), hawk–dove (HD), stag hunt (SH), and prisoner's dilemma (PD) games. Here the gray boxes indicate Nash equilibria. The right plot shows the phase diagram (preferred Nash equilibria) for the same region of payoff parameters.

that summarizes the individual incentives (increases of payoff for the active players) along the trajectories of the flow graph. In fact, the symmetric potential matrix ($V_{ij} = V_{ji}$) is defined by the following linear relationships:

$$V_{kj} - V_{ij} = A_{kj} - A_{ij} \quad \text{and} \quad V_{jk} - V_{ji} = A_{jk} - A_{ji} \tag{5}$$

between the components of the matrices **A** and **V**. These relationships remain valid for $n > 2$ if the existence criteria are satisfied (for details see Refs. [15,12]). Furthermore, if an N-player game (with equivalent players) is composed of pair interactions via uniform two-player potential games then the potential for the whole system can be given as

$$U(\mathbf{s}) = \sum_{\langle x,y \rangle} \mathbf{s}_x \cdot \mathbf{V} \mathbf{s}_y \tag{6}$$

where the summation runs over the interacting pairs ($x, y = 1, \ldots, N$) and $\mathbf{s} = (\mathbf{s}_1, \ldots, \mathbf{s}_N)$ denotes the strategy profile (or the microscopic state in the terminology of statistical physics). In these systems the payoff $\tilde{u}(\mathbf{s}_x)$ of player x comes from games with all the co-players at sites $x + \delta$, namely,

$$\tilde{u}_x(\mathbf{s}_x) = \sum_{\delta} \mathbf{s}_x \cdot \mathbf{A} \mathbf{s}_{x+\delta}, \tag{7}$$

and the variation of $\tilde{u}_x(\mathbf{s}_x)$, when the player modifies her strategy from $\mathbf{s}'_x$ to $\mathbf{s}''_x$, is equivalent to the variation of potential as

$$\tilde{u}_x(\mathbf{s}''_x) - \tilde{u}_x(\mathbf{s}'_x) = U(\mathbf{s}'') - U(\mathbf{s}') \tag{8}$$

where the strategy profiles $\mathbf{s}'$ and $\mathbf{s}''$ differ only for the player x, that is $\mathbf{s}_x = \mathbf{s}'_x$ for $\mathbf{s}'$ and $\mathbf{s}_x = \mathbf{s}''_x$ for $\mathbf{s}''$.

In evolutionary games the players can modify their own strategy by following a dynamical rule in order to receive higher individual payoffs. For example, the logit rule is defined by a random sequential strategy update with which player x modifies her strategy to $\mathbf{s}'_x$ with a probability

$$w(\mathbf{s}'_x) = \frac{e^{\tilde{u}_x(\mathbf{s}'_x)/K}}{\sum_{\mathbf{s}_x} e^{\tilde{u}_x(\mathbf{s}_x)/K}} \tag{9}$$

favoring exponentially the higher individual payoffs. The value of K quantifies the noise level. Blume [13] has shown that this logit rule drives the evolutionary potential games into the Boltzmann distribution, thus the probability of the strategy profile **s** in the final stationary state is

$$p(\mathbf{s}) = \frac{1}{Z} e^{U(\mathbf{s})/K} \tag{10}$$

where

$$Z = \sum_{\mathbf{s}} e^{U(\mathbf{s})/K}. \tag{11}$$

According to this result the strategy profile **s**, for which the potential reaches its maximum value has a distinguished role (in the analogy of ground state in many-particle systems). This feature can be exploited when evaluating the phase diagram on the T–S plane in the limit $K \to 0$ [35]. For the two-player game the CC strategy pair is preferred if $S > T - 1$ and $T < 1$ as it is illustrated on the right plot of Fig. 1. The CC strategy pair is the only Nash equilibrium for the harmony game while the maximum criterion prefers CC to DD in the mentioned region of the stag hunt game. Within the opposite region of the

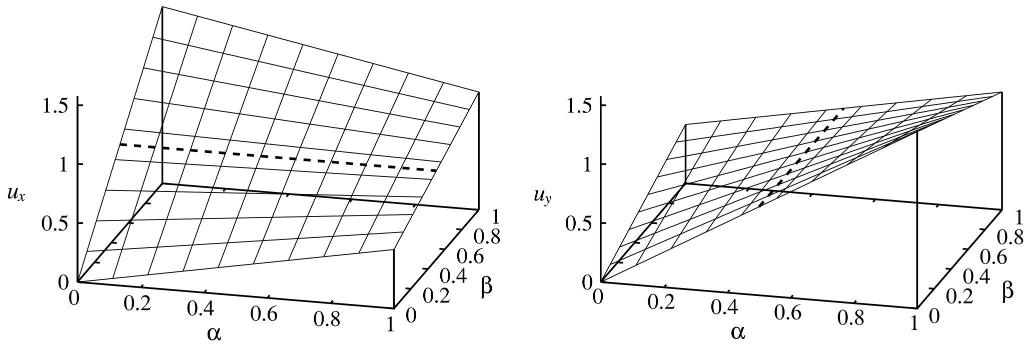

Fig. 2. Payoffs of the players x (upper) and y (lower) if they play a hawk–dove game when using mixed strategy with parameters α and β. The constant payoffs for both players are indicated by thick dashed lines if their opponent chooses the mixed Nash equilibrium with $\beta = \beta^\star$ and $\alpha = \alpha^\star$.

stag hunt game and also for the prisoner's dilemma DD is preferred because here $\max(V_{ij}) = V_{11}$. As $\max(V_{ij}) = V_{12} = V_{21}$ for the hawk–dove games therefore two equivalent preferred Nash equilibria exist here.

The mentioned preferences are inherited for most of the spatial models, thus these systems evolve into the homogeneous C (D) strategy state (in the low noise limit) within the white (dark) regions of the phase diagram (see the right plot of Fig. 1). In the hawk–dove region the preferred Nash equilibria can be realized on those lattices which can be divided into two sublattices in a way that the neighbors of a player belong to the opposite sublattice.

2.2. Mixed strategies

The mixed strategies for the players x and y are defined as

$$\mathbf{s}_x = \begin{pmatrix} 1-\alpha \\ \alpha \end{pmatrix} \quad \text{and} \quad \mathbf{s}_y = \begin{pmatrix} 1-\beta \\ \beta \end{pmatrix} \tag{12}$$

where α and β ($0 < \alpha, \beta < 1$) quantify the probabilities of choosing C for the players x and y. Their incomes, when playing a two-strategy game (3) against each other, are defined by Eqs. (2) and obey the following forms as a function of α and β:

$$\begin{aligned} u_x(\alpha, \beta) &= \alpha\beta + T(1-\alpha)\beta + S\alpha(1-\beta), \\ u_y(\alpha, \beta) &= \alpha\beta + S(1-\alpha)\beta + T\alpha(1-\beta). \end{aligned} \tag{13}$$

Within this set of strategies there exists an additional mixed Nash equilibrium as illustrated in Fig. 2 for a hawk–dove game if $T = 1.5$ and $S = 0.5$. The present (non-strict) Nash equilibrium is characterized by the parameters $\alpha^\star$ and $\beta^\star$ with values determined by the following equations:

$$\frac{\partial u_x(\alpha, \beta)}{\partial \alpha} = 0 \quad \text{and} \quad \frac{\partial u_y(\alpha, \beta)}{\partial \beta} = 0. \tag{14}$$

Straightforward calculations give that

$$\alpha^\star = \beta^\star = \frac{S}{T + S - 1} \tag{15}$$

and

$$u_x(\alpha, \beta^\star) = u_y(\alpha^\star, \beta) = \frac{ST}{T + S - 1} \tag{16}$$

in agreement with the results plotted in Fig. 2. Eq. (15) defines real solutions (with $0 < \alpha^\star = \beta^\star < 1$) within the regions of the hawk–dove and stag hunt games. There is, however, a relevant difference between the mixed Nash equilibria occurring in the mentioned regions of games. Namely, the mixed Nash equilibrium (with a suitable α) is an evolutionarily stable strategy for the hawk–dove games. This concept was introduced in the literature of theoretical biology [27] and specifies that a mutant cannot spread in a well-mixed population if its composition is equivalent to an evolutionarily stable state. In the present notation the mixed Nash equilibrium is an evolutionarily stable strategy if $u_y(\alpha^\star, \beta) \leq u_x(\alpha^\star, \alpha^\star)$ and $u_y(\alpha^\star, \beta) < u_x(\alpha^\star, \beta)$.

Similar stability criterion can be deduced for the dynamical games when the evolution of the well-mixed population of strategies is described by differential equations favoring the extension of successful strategies (species) having higher income (fitness) in the Darwinian competition [36]. The latter approach predicts fixed points for the compositions corresponding to mixed Nash equilibria. The stability analyses of these fixed points, however, have clearly indicated that these fixed points are attractive for the evolutionarily stable strategies and repulsive in the region of the stag hunt games.

The consequences of imitations in the payoff variations are also illustrated by Fig. 2. Namely, for the hawk–dove games the income of the active player is reduced by the imitation of a pure strategy and this loss vanishes if an evolutionarily stable strategy is imitated.

2.3. Three-strategy games

In the three-strategy model the players can choose one of the C, D, or mixed (M) strategies characterized by a parameter α. These strategies are denoted by three-dimensional unit vectors in the analogy of expressions (1), that is,

$$\mathbf{s}_x = D = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \text{or} \quad C = \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \text{or} \quad M = \begin{pmatrix}0\\0\\1\end{pmatrix}. \tag{17}$$

In this model the third mixed strategy is considered as a pure strategy. This approach can be interpreted in two ways. In the first case player x represents a group of individuals who all follow the same pure strategy (for C or D), or α and $(1-\alpha)$ portions of players use pure C and D strategies. The average income of each group comes from games between the pairs of individuals belonging to two neighboring groups. In the second case one can assume that the average payoffs of the players come from n ($n \gg 1$) consecutive games with using a pure or mixed strategy in the interactions with all their neighbors. For the evolutionary game we should assume additionally that the strategy refreshment is rare and we can neglect the variations of the average payoffs in the transient period after the strategy updates.

The payoffs are defined by a product as in Eqs. (2) and the corresponding payoff matrix is described by three parameters as

$$\mathbf{A} = \begin{pmatrix} 0 & T & \alpha T \\ S & 1 & \alpha + (1-\alpha)S \\ \alpha S & \alpha + (1-\alpha)T & \alpha^2 + \alpha(1-\alpha)(T+S) \end{pmatrix}. \tag{18}$$

The analysis of this system is simplified by the fact that this three-strategy game is also a potential game, as the payoff components satisfy the following relation [37,35,12]:

$$A_{12} - A_{21} + A_{23} - A_{32} + A_{31} - A_{13} = 0. \tag{19}$$

The symmetric potential matrix

$$\mathbf{V} = \begin{pmatrix} 0 & S & \alpha S \\ S & 1+S-T & S+\alpha(1-T) \\ \alpha S & S+\alpha(1-T) & \alpha^2(1-T-S)+2\alpha S \end{pmatrix} \tag{20}$$

satisfies the relationship (5) between **A** and **V**.

In the knowledge of the potential matrix one can evaluate the preferred pure Nash equilibrium and phase diagram in the zero noise limit by identifying the largest potential component ($\max(V_{ij})$), as above, for any values of α. This calculation has justified that the resultant phase diagram is identical to those (see the right plot of Fig. 1) we obtained for the two-strategy game. In words, a mixed strategy cannot be a preferred Nash equilibrium, therefore these are missing (independently of the values of α, T, and S) in the zero noise limit if logit rule controls the strategy updates. The latter observation is a direct consequence of two features. First, the payoff for a mixed strategy varies linearly between two limit values, thus it cannot be the largest one as it is illustrated graphically in Fig. 2. Furthermore, the rank of payoff values in each column of the payoff matrix agrees with the rank of potential components in the same column. Additionally we emphasize here that $V_{ij} = V_{ji}$ for the symmetric games therefore the ranks of potential values are equal in the ith row and column.

3. Monte Carlo simulations

Monte Carlo simulations are performed on a square lattice of $N = L \times L$ sites with periodic boundary conditions. The linear size of the system is varied from $L = 300$ to 1000. During the simulations we have determined the average values of strategy frequencies by averaging over a sampling time t_s after a relaxation time t_r. The values of t_s and t_r are varied from 10^4 to 10^5 Monte Carlo steps (MCS) where within 1 MCS each player has received a chance to modify her strategy once on average. The larger system sizes and the longer run times are used in the close vicinity of the critical phase transitions in order to suppress the undesired effects of the diverging fluctuations, correlation lengths, and relaxation time. Additionally we have used different initial states to achieve results characterizing the system behavior in final stationary state if $L \to \infty$.

The effects of the stochastic dynamics on the final stationary states are investigated for two types of evolutionary rules. In the first cases we applied the logit rule that can create sublattice ordered strategy arrangements. For the quantitative analysis of the latter structure the square lattice is divided into two sublattices in the analogy of the checker board and the strategy frequencies are determined in both sublattices.

In the second series of Monte Carlo simulations the evolution of strategy distribution is governed by a random pairwise strategy imitation [6]. In that cases we have repeated the following elementary steps when only one randomly selected

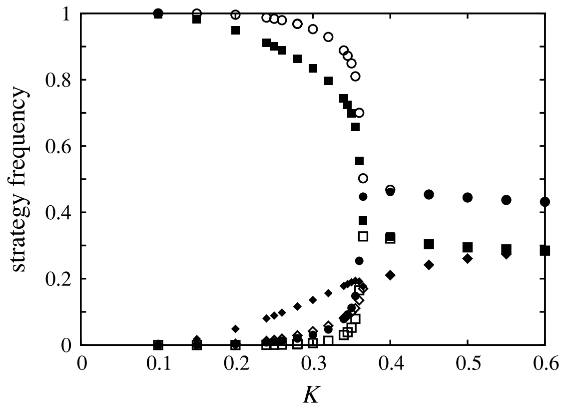

Fig. 3. Monte Carlo results for the strategy frequencies in the sublattices as a function of noise level K for $T = 1.3$, $S = 0.6$, and $\alpha = 0.5$. The boxes, bullets, and diamond indicate the frequencies of D, C, and M strategies, respectively. The open and closed symbols refer to data in the first and second sublattices.

player (at site x) is allowed to modify her strategy by imitating the strategy of another randomly selected neighboring player (at site y) with a probability

$$W(\mathbf{s}_x \leftarrow \mathbf{s}_y) = \frac{1}{1 + e^{(\tilde{u}_x(\mathbf{s}_x) - \tilde{u}_y(\mathbf{s}_y)/K)}} \tag{21}$$

where K measures the strength of stochastic events. This evolutionary rule has some well-known features. The strategy of the better neighbor is always adopted in the limit $K \to 0$. The three homogeneous states are absorbing states and these systems can exhibit critical phase transitions (belonging to the directed percolation universality class) between and absorbing states and strategy coexistence when tuning the payoff or noise parameters [38–40,2].

3.1. Results for logit rule

The Monte Carlo simulations (with applying a logit rule in the low noise limit) have confirmed the phase diagram (see the right plot of Fig. 1) derived theoretically.

If a homogeneous (D or C) phase is preferred at low noises then the simulations indicated a smooth decrease in the frequency of the dominating strategy when K is increased while the frequencies of the other two strategies are increased monotonously. Evidently, in the limit $K \to \infty$ all the three strategies are present with the same frequency and without any spatial correlations.

Critical phase transitions could be observed within the region of hawk–dove game as it is illustrated in Fig. 3. In these cases the system has two equivalent sublattice-ordered strategy arrangements on the square lattice for low noise levels in the analogy of the anti-ferromagnetic Ising model. Fig. 3 illustrates that the frequencies of the C and D strategies in the opposite sublattices decrease monotonously while the other two strategies occur with different frequencies in these ordered strategy distributions. In the disordered phase ($K > K_c$) the difference between the two sublattices disappear and the three strategies are present with the same frequencies in both sublattices. The plotted data refer to a critical phase transition belonging to the Ising universality class in agreement with the results obtained in similar systems [37,35,25]. More precisely, when approaching the critical point (K_c) from below then the differences of the strategy frequencies for both the C and D strategies in the two sublattices vanish algebraically and this transition is accompanied with a huge increase in the fluctuations, relaxation time, and correlation length.

In the region of stag hunt game the two-strategy version of the present model is equivalent to a ferromagnetic Ising model with an external magnetic field [19–24,12]. The systematic analyses of the kinetic Ising models [17,18] have indicated clearly the existence of meta-stable states represented by the permanent magnets. Similar phenomena can occur in the present evolutionary game even for three strategies. For example, if $S = -0.8$ and $T = 0$ then the "thermodynamically stable state" is the homogeneous cooperation ($s_x = C$ for all x) in the limit $K \to 0$. Despite it, the homogeneous defection (as the other pure Nash equilibrium in the lattice model) with some point defects can be observed in the Monte Carlo simulations for a long time at sufficiently low noise levels in agreement with the theoretical predictions of ordering processes [41]. Finally we mention that here the mean-field theory predicts an additional unstable solution resembling the mixed Nash equilibrium in this region of the S–T plane.

3.2. Results for imitation

The numerical investigations for imitations are restricted to a fixed noise level $K = 0.3$. This noise level was high enough to reduce the technical difficulties due to the slow relaxations at low noises. At the same time the results have indicated

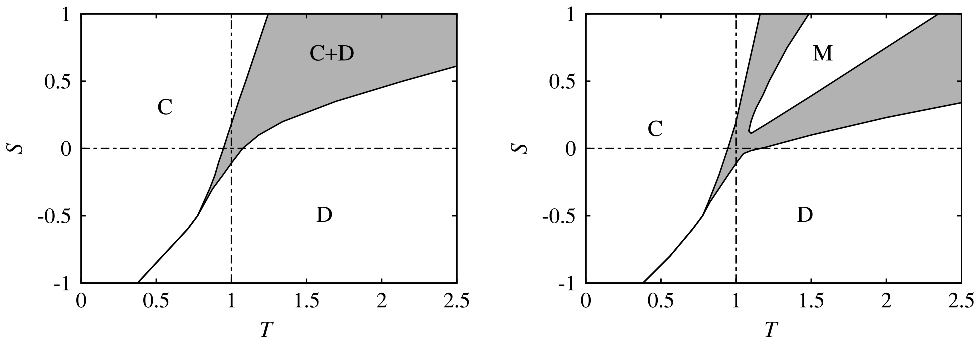

Fig. 4. Phase diagrams for the two-strategy (left) and three-strategy (right) systems on the S–T plane for imitation dynamics if $K = 0.3$. The white territories refer to homogeneous absorbing states while two or three strategies coexist in the gray regions.

the most relevant properties of this system at low values of K. Additionally, for the sake of simplicity $\alpha = 0.5$ is chosen for strategy M in all the simulations. For this value of α M is an evolutionarily stable strategy along the line $S = T - 1$ in the hawk–dove region of the S–T plane.

First we compare the resulting phase diagram (see the left plot of Fig. 4) for only two strategies (C and D) with the one we obtained for the application of the logit rule (right plot of Fig. 1) in the low noise limit. Notice that the territories of the homogeneous C and D phases are extended at the expense of the coexisting cooperative and defective strategies in the hawk–dove region of payoffs. Hauert and Doebeli have shown that in this region of coexistence the portions of C and D strategies vary smoothly without any sign of sublattice ordering. Here the imitations of C or D strategies are disadvantageous for both the masters and followers. Consequently the disordered strategy arrangements change continuously. In the region of stag hunt game the shift of the $C-D$ phase boundary quantifies the advantage of imitation in the maintenance of cooperation on lattices [42].

In the right plot of Fig. 4 white territories illustrate the locations of all the three absorbing states on the S–T plane. Notice that the boundaries of the homogeneous C and D phases are slightly shifted due to the presence of M players during the evolutionary competition. At the same time the $C - D$ phase boundary is not changed in the stag hunt region because M strategies become extinct within a short transient period and afterwards the system behaves like a two-strategy model.

The most striking consequence of the imitation rule is the appearance of the homogeneous M phase in the hawk–dove region instead of the sublattice ordered spatial distribution of C and D players. Here we have to emphasize that the present imitation rule is not capable of forming a sublattice ordered arrangement of two different strategies. Furthermore, the investigations of the two-strategy version of this system have indicated the coexistence of the C and D players without any sign of sublattice ordering in previous investigations [43]. The imitations of C or D strategies are disadvantageous for both the masters and followers. The imitation caused loss in payoff can vanish when M is imitated (see Fig. 2). Consequently, M dominates the system behavior in the vicinity of payoff regions where it is an evolutionarily stable strategy.

In order to have a more complete picture of what happens in the regions of coexistence (gray territories in the $S - T$ phase plotted by Fig. 4) we show the variation of strategy frequencies when T is increased for two fixed values of S. The left plot of Fig. 5 shows the Monte Carlo results we obtained for $S = 0.3$. Here the statistical error is smaller than the line thickness, therefore the strategy frequencies are illustrated by three different line types. The plotted results show clearly the main characteristic of the extinction of the vanishing strategy when approaching one of the critical points where only one strategy remains alive. In principle, for all the four critical transitions the vanishing strategy exhibits a power law behavior with the same critical exponent characteristic of the directed percolation universality class. The plotted numerical data are consistent with the theoretical prediction. In this example only two strategies exist in the regions of coexistence. On the contrary, the right plot of Fig. 5 illustrates the existence of stationary states with three strategies.

According to the principle of universality the extinction of the third strategy (M) within the coexistence region also possesses the general features of the directed percolation universality class. The numerical justification of this expectation, however, goes beyond the scope of the present study.

4. Summary

We have studied what happens in a multi-agent two-strategy evolutionary game if the players are allowed to use mixed strategies in the pair-interactions with their neighbors. In the extended model the third strategy is considered as an additional pure strategy with suitable payoffs. It is found that the resulting 3×3 matrix game is a potential game, if the players are equivalent, that is, if the original two-strategy matrix game is symmetric. We have evaluated the potential matrices for both the 2×2 and 3×3 pair interactions. In the knowledge of potential matrices we could identify the preferred Nash equilibria in both the two-player systems and the multi-player evolutionary games if the players are located on the sites of a square lattice. The comparison of the preferred Nash equilibria has clearly indicated the irrelevance of the mixed

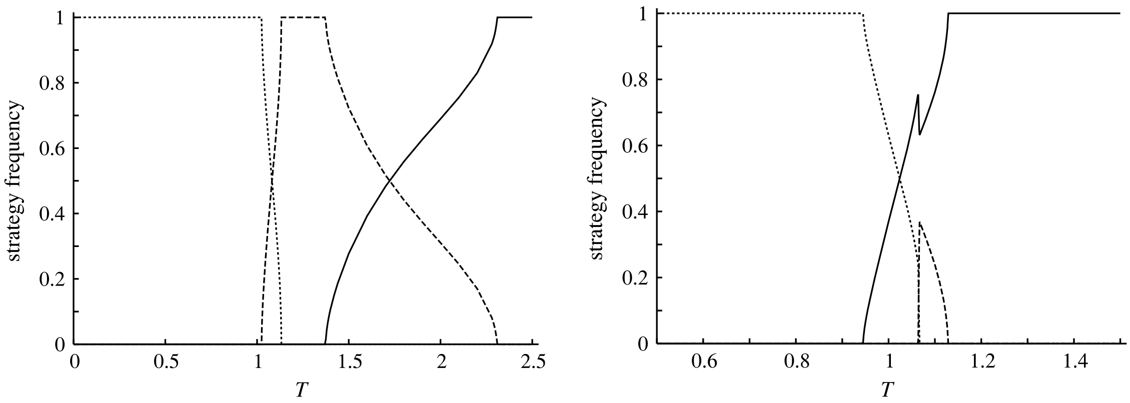

Fig. 5. Strategy frequencies versus T for imitation if $K = 0.3$, and $\alpha = 0.5$ at $S = 0.3$ (left) and $S = 0$ (right). The solid, dotted and dashed lines illustrate the frequencies of D, C, and M strategies.

strategy in those multi-agent evolutionary games where the dynamics is governed by a logit rule at low noise levels. These results are confirmed by Monte Carlo simulations for a wide scale of payoffs.

Using Monte Carlo simulations we have observed a significantly different behavior in the above systems if the strategy evolution is based on imitating a better neighbor. In agreement with the results of population dynamics the mixed strategy M has dominated the system behavior on those region of payoffs where M is an evolutionarily stable strategy. In agreement with previous analyses we have observed striking difference in the behavior when comparing the results caused by two different evolutionary dynamics. This difference is related to the fact that the imitation does not support the creation of a sublattice ordered strategy arrangement that may easily occur for the application of the logit rule on a square lattice. On the other hand, the imitation of an evolutionarily stable strategy is not accompanied with a payoff decrease and this fact increases the probability that a mixed strategy will be imitated in the next steps.

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References

[1] M.A. Nowak, Evolutionary Dynamics, Harvard University Press, Cambridge, MA, 2006.
[2] G. Szabó, G. Fáth, Evolutionary games on graphs, Phys. Rep. 446 (2007) 97–216.
[3] B. Allen, M.A. Nowak, Games on graphs, EMS Surv. Math. Sci. 1 (2014) 113–151.
[4] J. von Neumann, O. Morgenstern, Theory of Games and Economic Behaviour, Princeton University Press, Princeton, 1944.
[5] M.A. Nowak, R.M. May, The spatial dilemmas of evolution, Internat. J. Bifur. Chaos 3 (1993) 35–78.
[6] G. Szabó, C. Tőke, Evolutionary prisoner's dilemma game on a square lattice, Phys. Rev. E 58 (1998) 69–73.
[7] G.W. Brown, Iterative solution of games by fictious play, in: T.C. Koopmans (Ed.), Activity Analysis of Production and Allocation, Wiley, New York, 1951, pp. 373–376.
[8] D. Monderer, L.S. Shapley, Fictious play property for games with identical interests, J. Econom. Theory 68 (1996) 258–265.
[9] M. Sysi-Aho, J. Saramäki, J. Kertész, K. Kaski, Spatial snowdrift game with myopic agents, Eur. Phys. J. B 44 (2005) 129–135.
[10] W.H. Sandholm, Pairwise comparison dynamics and evolutionary foundations for nash equilibrium, Games 1 (2010) 3–17.
[11] G. Szabó, A. Szolnoki, Selfishness, fraternity, and other-regarding preference in spatial evolutionary games, J. Theoret. Biol. 299 (2012) 81–87.
[12] G. Szabó, I. Borsos, Evolutionary potential games on lattices, Phys. Rep. 624 (2016) 1–60.
[13] L.E. Blume, The statistical mechanics of strategic interactions, Games Econom. Behav. 5 (1993) 387–424.
[14] L.E. Blume, The statistical-mechanics of best-response strategy revision, Games Econom. Behav. 11 (1995) 111–145.
[15] D. Monderer, L.S. Shapley, Potential games, Games Econom. Behav. 14 (1996) 124–143.
[16] W.H. Sandholm, Population Games and Evolutionary Dynamics, MIT University Press, Cambridge, MA, 2010.
[17] C. Domb, Ising model, in: C. Domb, M.S. Green (Eds.), Phase Transitions and Critical Phenomena, Vol. 3, Academic Press, London, 1974, pp. 357–484.
[18] R.J. Glauber, Time-dependent statistics of the Ising model, J. Math. Phys. 4 (1963) 294–307.
[19] S. Galam, Y. Gefen, Y. Shapir, Sociophysics: A mean behavior model for the process of strike, Math. J. Sociol. 9 (1982) 1–13.
[20] A.V.M. Herz, Collective phenomena in spatially extended evolutionary games, J. Theoret. Biol. 169 (1994) 65–87.
[21] W.A. Brock, S.N. Durlauf, Discrete choice with social interactions, Rev. Econom. Stud. 68 (2001) 235–260.
[22] G. Weisbuch, D. Stauffer, "Antiferromagnetism" in social relations and Bonabeau model, Physica A 384 (2007) 542–548.
[23] S. Galam, B. Walliser, Ising model versus normal form game, Physica A 389 (2010) 481–489.
[24] S. Grauwin, D. Hunt, E. Bertin, P. Jensen, Effective free energy for individual dynamics, Adv. Complex Syst. 14 (2011) 529–536.
[25] G. Szabó, K.S. Bodó, B. Allen, M.A. Nowak, Four classes of interactions for evolutionary games, Phys. Rev. E 92 (2015) 022820.
[26] K. Sigmund, The Calculus of Selfishness, Princeton University Press, Princeton, NJ, 2010.
[27] J. Maynard Smith, Evolution and the Theory of Games, Cambridge University Press, Cambridge, UK, 1982.
[28] J. Maynard Smith, G.R. Price, The logic of animal conflict, Nature 246 (1973) 15–18.
[29] K. Sigmund, Game dynamics, mixed strategies and gradient systems, Theor. Popul. Biol. 32 (1987) 114–126.
[30] J. Berg, M. Engel, Matrix games, mixed strategies, and statistical mechanics, Phys. Rev. Lett. 81 (1998) 4999–5002.
[31] C.E. Tarnita, T. Antal, M.A. Nowak, Mutation-selection equilibrium in games with mixed strategies, J. Theoret. Biol. 261 (2009) 50–57.
[32] K. Miyaji, J. Tanimoto, A. Hagishama, N. Ikegaya, Influence of stochastic perturbation of both action updating and strategy updating in mixed-strategy 2×2 games on evolution of cooperation, Phys. Rev. E 88 (2013) 062149.

[33] J. Tanimoto, Correlated asynchronous behavior updating with a mixed strategy system in spatial prisoner's dilemma games enhances cooperation, Chaos Solitons Fractals 80 (2015) 39–46.
[34] S. Kokubo, Z. Wang, J. Tanimoto, Spatial reciprocity for discrete, continuous and mixed strategy setups, Appl. Math. Comput. 259 (2015) 552–568.
[35] J. Vukov, L. Varga, B. Allen, M.A. Nowak, G. Szabó, Payoff components and their effects in a spatial three-strategy evolutionary social dilemma, Phys. Rev. E 92 (2015) 012813.
[36] R. Cressman, Evolutionary Dynamics and Extensive Form Games, MIT Press, Cambridge, MA, 2003.
[37] G. Szabó, K.S. Bodó, B. Allen, M.A. Nowak, Fourier decomposition of payoff matrix for symmetric three-strategy games, Phys. Rev. E 90 (2014) 042811.
[38] P. Grassberger, On phase-transitions in Schlögl 2nd model, Z. Phys. B 47 (1982) 365–374.
[39] J. Marro, R. Dickman, Nonequilibrium Phase Transitions in Lattice Models, Cambridge University Press, Cambridge, UK, 1999.
[40] H. Hinrichsen, Non-equilibrium critical phenomena and phase transitions into absorbing states, Adv. Phys. 49 (2000) 815–958.
[41] A.J. Bray, Theory of phase ordering kinetics, Adv. Phys. 43 (1994) 357–459.
[42] M.A. Nowak, Five rules for the evolution of cooperation, Science 314 (2006) 1560–1563.
[43] C. Hauert, M. Doebeli, Spatial structure often inhibits the evolution of cooperation in the snowdrift game, Nature 428 (2004) 643–646.