Phase-transition study of a one-dimensional probabilistic site-exchange cellular automaton

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The effect of mixing on a one-dimensional probabilistic cellular automaton with totalistic rule has been investigated by different methods. The evolution of the system depends on two parameters, the probability p and the degree of mixing m. The two-dimensional phase space of parameters has been explored by simulation. The results are compared to the multiple-point-correlation approximation. By increasing the mixing, the order of the phase transition has been found to change from second order to first order. The tricritical point has been located and estimates are given for the β exponent. Short- and long-range mixing are compared.

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I. INTRODUCTION

The effect of mixing between subsequent synchronous cellular automata (CA) iterations has been studied intensively recently [1–4]. The interest comes from the fact that for strong mixing the correlations created by the automata rule "are washed out" and one expects the recovery of the mean-field result. Those models are especially interesting when the mixing causes changes of the phase structure. Phenomenologically it corresponds to infection models in theoretical epidemiology.

We have studied the model that was earlier investigated by Bidaux *et al.* [5] in several dimensions. Their conclusion was that for one dimension the phase transition is continuous contrary to the prediction of mean-field approximation, however, one can observe the emergence of a first-order transition for higher dimensions. This finding is analogous to the phase-transition phenomena of equilibrium statistical physics.

Now we attack this correspondence theorem from two directions: introducing sequential mixing between subsequent CA updates and upgrading the mean-field calculations by taking into account multiple-point correlations. While the first method destroys correlations in a simulation, pushing behavior towards the mean field, the second method incorporates correlations, pushing the behavior from the mean field (one-point-correlation approximation) towards the real simulated behavior .

Locating the crossover from a second-order to a first-order transition is a difficult task. We have attempted it by calculating the order-parameter critical exponent.

Two different kinds of mixing mechanism were simulated. A short-range diffusionlike mechanism, which allows nearest-neighbor hopping, and a long-range one, which permits jumping of a live cell anywhere in the system.

II. THE MODEL

The one-dimensional (1D), two-state ($\{0,1\}$) probabilistic cellular automaton is defined by two subrules: (a) a totalistic, synchronous, range-4 rule,

$$s(t+1,j)=\begin{cases} X & \text{if } 3\le\sum_{j-4}^{j+4}s(t,j)\le 6 \\ 0 & \text{otherwise}\end{cases}$$

where $X\in\{0,1\}$ is a two-valued random variable such that

$$\mathrm{Prob}(X=1)=p;$$

and (b) a sequential, site-exchange rule that $m\times N$ [$m\times$ (number of living cells) (1)] are selected randomly and jumped to empty (0) sites between subsequent sweeps of the first subrule.

The destination place can be nearest-neighbor site (short range) or any site of the lattice (long range).

The deterministic case ($p=1$) exhibits class-3 CA behavior [6], class-3 behavior being characterized by chaotic evolution towards an attractor, which is most often strange. For a large number of iterations of the two subrules, the system evolves towards either an empty state, where the concentration (c) of 1's is zero, or towards some finite-concentration state. Regarding c as the order parameter, the phase transition occurs at some $p_c(m)$.

The $m = 0$ and $m = \infty$ cases were investigated by [5]. For $m = 0$ they found the phase transition to be of second order, however the universality class differs from that of directed percolation.

The mean-field approximation corresponds to $m = \infty$, which exhibits first-order phase transition. This implies that there must be a crossover point for some finite value of m.

The purpose of our work is to explore the $p(m)$ phase space and locate the tricritical point for short- and long-range mixing.

III. SIMULATIONS

The simulations were done on an $L = 40\,000$ lattice, after checking with $L = 100\,000$ lattice simulations that the finite-size effect does not affect the results within statistical error. The numerical procedure is similar to that of [1], [2], and [3].

For fixed values of m the steady-state concentrations $c(m, p, \infty)$ were determined within the neighborhood of the $p_c(m)$ phase-transition point. The critical scaling behavior of the order parameter can be characterized by the $\beta(m)$ exponent:

$$\beta(m) = \lim_{p \to p_c(m)+0} \frac{\ln c(m, p, \infty)}{\ln[p - p_c(m)]}.$$

The exponent β and p_c are calculated simultaneously by least-squares-error fitting to the above formula. The least-squares error was minimized for 1000 trial $p_c(m)$'s.

The quality of the p_c estimate is checked by an independent "extinction method" fitting [1] for $m = 0$ and $m = 3$ too. This method is based on measuring the time (t) required to fall into the absorbing state from a randomized initial state of $c = 0.1$ below the transition point $(p_c(m))$, and exploiting the scaling behavior of it:

$$t \sim [p - p_c(m)]^{-\nu}.$$

The simulation is done for each m by determining 15–30 $\{p(m), c(m, p)\}$ points following 40 000–150 000 iterations. The steady-state concentrations are averaged values of 10 000 iterations. The final state of the system for a given p is taken to be the initial state for the next run. This permits faster convergence to the steady state. The typical run time on SUN 4 workstations is about one day per point, although it depends on m strongly.

We have obtained a similar shape of $p_c(m)$ curves for long- and short-range mixing although the scales are different (Figs. 1 and 2). Introducing m to the mixing-free model investigated by Bidaux *et al.* [5] the p_c increases from $p_c \simeq 0.721$. It approaches a maximum value of $p_c \simeq 0.7341$ at $m = 0.05$ for long-range and $p_c \simeq 0.7594$ at $m \simeq 12$ for short-range mixing. Then the curves turn back and converge to the mean-field value, $p_c \simeq 0.5347$, as $m \to \infty$. The effect of long-range mixing is much stronger, as expected.

The numerical estimates for β are somewhat less accurate, because of the great sensitivity to p_c. We expect

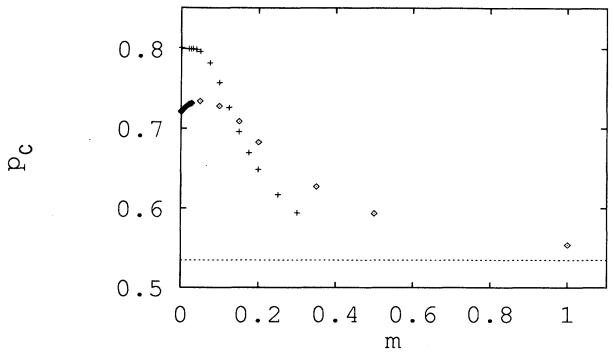

FIG. 1. Phase space $p_c(m)$ for long-range mixing. Empty squares: simulation result. Crosses: four-point-correlation result. The dashed line corresponds to the mean-field value $p_c(m = \infty) = 0.5347$.

first-order phase-transition behavior by simulation data when the $p_c(m)$ corresponding to lowest error fitting is smaller than the highest zero concentration steady state $p(m)$. In other words the power-law behavior of second-order transition breaks down, and hysteresis appears as a natural sign of first-order transition.

For long-range mixing this occurs at $m \simeq 0.015$, when the β exponent has fallen to $\beta_t = 0.17 \pm 0.01$ about two-thirds of the critical transition universality value $\beta_c = 0.28$ (Fig. 3). Note that we may expect roundings on the order-parameter curve owing to finite size effects too.

We have checked the effect of the finite size near the crossover point too. Simulations for $m = 0.01$ and $m = 0.02$ showed that the difference between the $L = 40\,000$ and $L = 60\,000$ lattices results in p_c is below 0.05%, much less than the statistical error. The β exponents differed by 1% only, and while for $m = 0.01$ the transition is second order it is first order for $m = 0.02$ already, in agreement with the $L = 40\,000$ lattice results.

For short-range mixing the transition remains continuous for $m \leq 40$ with $\beta \simeq 0.28$ again (Fig. 4), which is about the universality class value of the 2D directed percolation model. The exponent β falls from $\beta(0) = 0.41$ to this constant value in the $0 < \beta < 0.0005$ region. This suggests the universality changes abruptly within this region. The crossover to first-order transition is above $m = 40$.

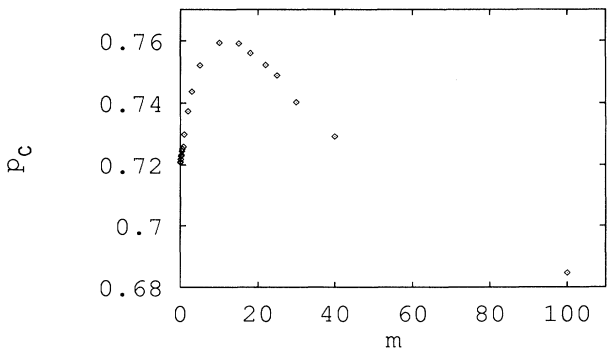

FIG. 2. Phase space $p_c(m)$ for short-range mixing. Simulation result.

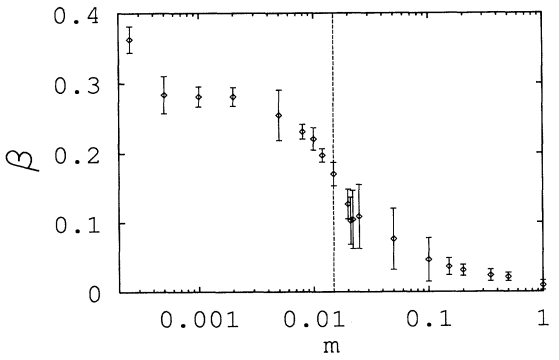

FIG. 3. Critical exponent β, obtained by fitting from simulated data for the long-range-mixing case. The transition is second order for $m < 0.015$. Tricritical point occurs at $m = 0.015$ (dashed line). Data points for $m > 0.015$ (first-order transition region) do not correspond to the β exponent, but characterize a steep jumping transition behavior only.

For $m = 100$ we could do simulations for $L = 10\,000$ and 20 000–40 000 iterations only because of the excessive CPU time. This shows that the transition is already first order there. In conclusion the crossover must happen between $m = 40$ and $m = 100$, but the huge CPU time requirement on the computers at our disposal excluded to locate the crossover point more precisely.

IV. MULTIPLE-POINT CORRELATIONS

The usual mean-field approximation assumes that field variables are affected by an averaged effect of the surroundings neglecting any existing correlations. In the cellular automaton model this background field is the concentration c of the "1" (live) cells. The mean-field analysis of this CA model has been done by Bidaux *et al.* [5] and first-order transition with $p_c \simeq 0.5347$ was found.

As it was pointed out by Gutowitz *et al.* [7] and applied in practice for 1D driven lattice-gas model [8], the effect of correlations can be taken into account by setting up

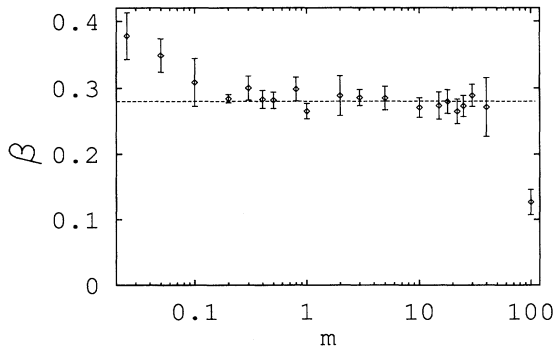

FIG. 4. Critical exponent β, obtained by fitting from simulated data for the short-range-mixing case. The transition is second order for $m \geq 40$. Tricritical point is above $m = 40$. The dashed line shows the percolation transition β exponent.

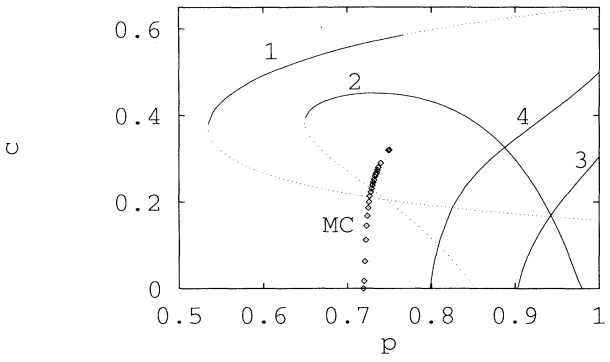

FIG. 5. Multiple-point correlation results for the phase transition of $m = 0$ mixing-free CA model. The numbers show the level of approximations. 1: mean field. Dotted lines correspond to unstable solutions. The symbols correspond to simulation result.

equations for two, three, or more neighboring-point configuration probabilities. By determining numerically the stable, steady-state solutions of these equations, one can express the $c(p)$ curves. The details of the method are similar to that of Szabó *et al.* [8].

The calculations have been done for the level of two-, three-, and four-point correlation. The calculated concentrations as a function of p are plotted in Fig. 5 for $m = 0$. This figure shows that p_c increases when taking the two-point correlation into account. Furthermore, p_c has a maximum for the three-point correlations. In this case the transition becomes second order. The transition point decreases toward the MC data and the transition remains continuous at the level of four-point approximation.

This nonmonotonic convergence of p_c towards the simulated p_c is in agreement with the $p_c(m)$ phase diagram (Figs. 1 or 2). The degree of correlation seems to be proportional to $1/m$, consequently, the $p_c(m)$ curve tends to $p_c \simeq 0.721$ by a similar nonmonotonic pattern if we start from $1/m = 0$ and follow along the line.

Fitting to the results of the four-point approximation gives $\beta = 0.446 \pm 0.002$, which is close to the simulated $\beta \simeq 0.41$. Thus, the n-point correlations seem to be not

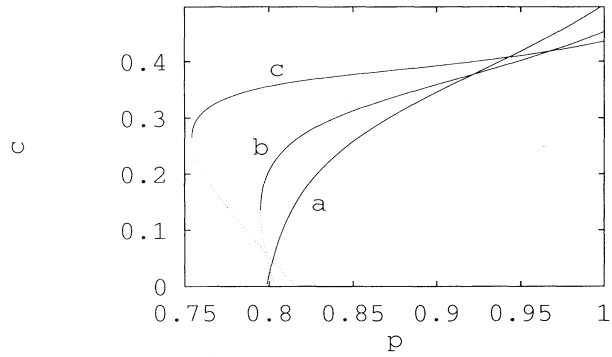

FIG. 6. Four-point correlation with long-range-mixing effect. a: $m = 0$ second order; b: $m = 0.05$ first order; c: $m = 0.1$ first order. Dotted lines correspond to unstable solutions.

relevant for $n > 4$.

The calculation mentioned above may be easily extended by taking the long-range-mixing effect into account at the level of four-point approximation. For this purpose we determined the contribution of the single hopping to the multiple-point configuration probabilities, which is valid for $m << 1$. By this means we could evaluate the average concentration as a function of p for different m. In agreement with the expectation Fig. 6 demonstrates that the continuous transition becomes first order when m increases. This method allows us to determine p_c vs m. The results agree well with the Monte Carlo data (see Fig. 1). The four-point approximation gives an estimate for the tricritical point of $m \simeq 0.025$.

V. CONCLUSIONS

The critical behavior of the probabilistic cellular automata model has been studied by using Monte Carlo simulations and adopting the dynamical mean-field theory taking the multiple-point correlations into account. The effect of correlations on the phase transition was investigated and has been confirmed by both methods to be essential. The order of the transition changes from second order to first order when eliminating the correlations. More precisely, the dynamical mean-field theory predicts first-order transition when the analysis is restricted to the effect of one- and two-point correlations. This theory suggests continuous transition if the three- and/or four-point correlations are not neglected.

In Monte Carlo simulations the correlations are washed out by introducing a mixing mechanism between two subsequent CA sweeps. The short- and long-range mixing mechanism has been compared and the crossover point is located. The approximately three orders of magnitude difference in the transition point between the short- and long-range mixing effects suggests that short-range mixing is very ineffective in destroying the strong correlations. By determining the critical exponent β the change of universality could be observed for very weak mixing. The two-parameter (p - m) phase structure has been explored by simulation and found to be in agreement with multiple-point-correlation calculation. While the weak mixing effect causes an increase of the transition p_c, stronger site exchange decreases it.

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