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Bursts in three-strategy evolutionary ordinal potential games on a square lattice

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HIGHLIGHTS

- Evolutionary potential games are perturbed by a cyclic component for logit rule.
- The cyclic component causes paradoxical effects and bursts in the spatial model.
- The bursts are related to two consecutive nucleation processes.
- The noise dependence of frequencies, payoffs and fluctuations are determined.

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ABSTRACT

Evolutionary potential games provide an analogy for equilibrium statistical physics under suitable conditions that is broken if cyclic components are added. Using Monte Carlo simulations we study the effect of weak cyclic perturbations on the equilibrium in a system of players located on a square lattice while the dynamics is controlled by a logit rule. The pair interaction is composed of a symmetric three-strategy coordination game of unit strength, a weak self-dependent term, and a cyclic (rock–paper–scissors) component. The self-dependent component favors the first (rock) strategy which dominates the system behavior at low noises if the strength of the cyclic component is below a threshold value. In the opposite case the predator of the first strategy (paper) rules the behavior at low noises while the first strategy can occasionally form growing domains (bursts) diminished after the appearance of the third one. The variation of noise modifies the invasion velocities as well as the rate of nucleation processes and yields drastic changes in the strategy frequencies, fluctuations, average payoffs, and spatio-temporal patterns.

1. Introduction

In evolutionary games equivalent players are located at the sites of a connectivity network or lattice [1,2] and the pair interactions between the neighboring players are described by an $n \times n$ payoff matrix. In these systems each player collects income via games with their neighbors. Additionally, the players are allowed to modify their strategy unilaterally by following a prescribed evolutionary rule. These models are widely used to study the final stationary state of biological, ecological, and economic systems.

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The early investigations of spatial evolutionary models [3] concentrated on two-strategy games where the interaction is well defined by only two parameters due to the presence of an irrelevant constant and the possibility of choosing a suitable payoff unit [4]. For these games different types of interactions and macroscopic behaviors are distinguished. For example, the equivalence between some evolutionary games and the Ising model [5,6] was discussed by several authors [7,8]. The existence of a potential matrix [8–11] guarantees that all these evolutionary games evolve into a Gibbs distribution if the so-called logit rule [12–17] controls the dynamical processes. This general feature is preserved for n-strategy potential games which do not contain the cyclic components [11,18,19]. For $n = 3$ there exists only one cyclic component, represented by the rock–paper–scissors game, and this fact simplifies the analysis of the interplay between the potential and cyclic components.

In ordinal potential games [9,11,20,21] the cyclic components are weak and not capable of modifying the preference of strategy choices for the possible unilateral changes. Consequently, the pure Nash equilibria of the potential component are inherited while the thermodynamical behavior (represented by the Gibbs distribution, detailed balance, ordered states at low noises, and phase transitions) is destroyed by the presence of cyclic components that generate probability currents in the dynamical graph [22].

Now we study a three-strategy evolutionary ordinal potential game on a square lattice in which the strategy update is governed by the logit rule. The equivalent pair interactions between neighbors are composed of three types of elementary games [19] (representing coordinations, external effects and cyclic dominance) as detailed in Section 2. The curious behavior of this model and the appearance of bursts in a region of parameter space will be investigated by Monte Carlo simulations (in Section 3) which show striking variations in the strategy frequencies, fluctuations, average payoffs, and spatio-temporal patterns, when varying the noise level and payoff parameters. Finally we summarize the phenomena and their general features which may occur in many other evolutionary games and systems.

2. The model and its general features

In the present model equivalent players are located on the sites (x) of a square lattice with a linear size of L. The undesired finite-size and boundary effects are suppressed by applying periodic boundary conditions for sufficiently large linear sizes. The traditional matrix notation is used for the quantitative description of the interaction between two neighboring players. Accordingly, for symmetric three-strategy games the pure strategies are defined by three-dimensional unit vectors as:

$$\mathbf{s}_x = \begin{pmatrix}1\\0\\0\end{pmatrix}, \quad \begin{pmatrix}0\\1\\0\end{pmatrix}, \quad \text{or} \quad \begin{pmatrix}0\\0\\1\end{pmatrix}, \tag{1}$$

and we assume that each player uses the same pure strategy against all of her/his nearest neighbors. Thus, the accumulated income $u_x(\mathbf{s}_x)$ of player x is given by

$$u_x(\mathbf{s}_x) = \sum_{\delta} \mathbf{s}_x \cdot \mathbf{A}\mathbf{s}_{x+\delta} \tag{2}$$

where the summation runs over the neighboring sites $x + \delta$, and the payoff matrix $\mathbf{A}$ is composed of three elementary components [19,23] as,

$$\mathbf{A} = \begin{pmatrix} 1+\varepsilon & \varepsilon-\lambda & \varepsilon+\lambda \\ \lambda & 1 & -\lambda \\ -\lambda & \lambda & 1 \end{pmatrix}. \tag{3}$$

For $\varepsilon = \lambda = 0$ this interaction is equivalent to those defined by the three-state Potts model with a unit coupling constant [24,25]. In the terminology of matrix decomposition [11,19] this interaction includes equivalent coordinations between all three strategy pairs. The parameter $\varepsilon > 0$ describes an external effect favoring the choice of the first strategy. The value $\lambda > 0$ quantifies the strength of the cyclic component in such a way that the first, second and third strategies correspond to the choice of rock (R), paper (P), and scissors (S) in the traditional game. Henceforth the strategy labels $i = 1, 2,$ and 3 refer to R, P, and S, respectively. Our analyses will be restricted to the small values of ε and λ when the corresponding two-player game has three pure Nash equilibria, namely, the R-R, P-P, and S-S strategy pairs. Due to the external support ($\varepsilon > 0$) the strategy pair R-R is the preferred Nash equilibrium.

In the evolutionary lattice model a randomly selected player (located at site x) is allowed to modify her/his strategy unilaterally from $\mathbf{s}_x$ to $\mathbf{s}'_x$ with a probability exponentially favoring higher individual incomes. More precisely, for this logit rule the probability of choosing strategy $\mathbf{s}'_x$ is expressed by

$$w(\mathbf{s}'_x) = \frac{e^{u_x(\mathbf{s}'_x)/K}}{\sum_{\mathbf{s}_x} e^{u_x(\mathbf{s}_x)/K}} \tag{4}$$

where K quantifies the frequency of mistakes in the decision processes and plays the role of temperature in physical systems.

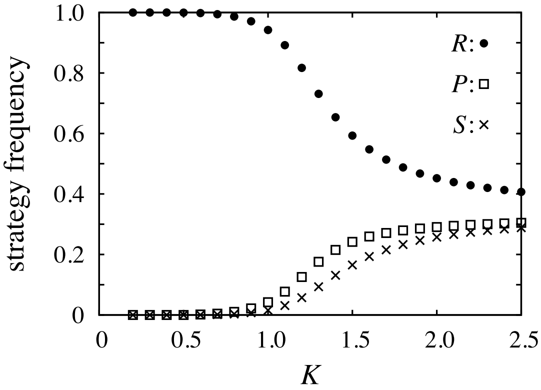

Fig. 1. Strategy frequencies versus noise for $\varepsilon = \lambda = 0.1$. Black bullets stand for strategy R, open boxes show the frequency of strategy P, and crosses denote strategy S.

For $\varepsilon = \lambda = 0$ this model has three equivalent ordered (homogeneous) states in the limit $K \to 0$. Consequently, if K is increased, then this system undergoes an order–disorder phase transition (at $K = K_c = 1/\ln(1+\sqrt{3}) = 0.995$ [24]) representing a universality class in the theory of phase transitions [25–27]. The external effect ($\varepsilon > 0$, $\lambda = 0$) destroys the equivalence of the three ordered states by preferring strategy R and the critical phase transition is smoothed out.

The cyclic component in itself has a mixed Nash equilibrium [28] and maintains the coexistence of the three strategies [2,16,29–33]. In the absence of the external field ($\varepsilon = 0$) the cyclic component prevents the formation of a long-range ordered (homogeneous) phase at low noise levels. Instead, the three strategies coexist with the same frequency and form large domains which are separated by rotating spiral arm interfaces [34]. Here we have to emphasize that similar self-organizing patterns are observed if stochastic imitations control the evolution of strategy distribution [35]. For imitation based rules the players can adopt (imitate) one of the neighboring strategies with a payoff dependent probability [2]. In this case strategy changes can only occur if the neighboring players choose different strategies. In other words, this evolutionary process supports the formation of homogeneous domains and prevents the appearance of single (isolated) foreign strategies inside the homogeneous domains, therefore the analysis of the interfacial topology is simplified [36]. Finally we mention that many versions of these systems were previously investigated using mean-field approximations [16,30,37–39], lattice models [2,40] or the complex Ginzburg–Landau approach [41–43].

In contrast to imitation, the logit rule allows players to choose a strategy not present in their neighborhood and this strategy can initiate the birth and expansion of a new domain as detailed in the next section.

3. Monte Carlo simulations

The interplay of the three types of interactions is investigated by Monte Carlo (MC) simulations when varying the parameters. During the simulations on a large lattice (the linear size is varied from $L = 600$ to 2000) we have recorded the strategy frequencies $\rho_i(t)$ as a function of time t, and also the average payoffs, $p_i(t)$, received by players following the ith strategy. In one time unit (MCS) each player had a chance to change her/his strategy once on average.

The average strategy frequencies $\bar{\rho}_i = \langle \rho_i(t) \rangle$ and their fluctuations $\chi_i = N \langle (\rho_i(t) - \bar{\rho}_i)^2 \rangle$ (with $N = L^2$), as well as the strategy specific average payoffs, $\bar{p}_i = \langle p_i(t) \rangle$, are obtained by averaging over a sampling time t_s after a relaxation time t_r. The average total payoff, $\bar{p} = \sum_i \bar{\rho}_i \bar{p}_i$, is also determined. The values of t_s and t_r are adjusted according to the behavior of the system for each point in the parameter space (10^4 MCS $< t_r, t_s < 5 \cdot 10^7$ MCS). In our MC data the statistical error is comparable to the symbol size.

For weak cyclic dominance the system behavior is ruled by the Potts component and the external field favoring strategy R at low noise levels. In these cases Fig. 1 illustrates the typical variation of strategy frequencies as a function of K. This plot shows a smooth transition from the homogeneous R state ($\bar{\rho}_R = 1$ and $\bar{\rho}_P = \bar{\rho}_S = 0$) towards the disordered strategy distribution ($\bar{\rho}_i = 1/3$) occurring in the limit $K \to \infty$. In order to help the readers the selected symbols in Fig. 1 resemble the strategies rock, paper, and scissors.

As mentioned above the two-dimensional system exhibits a self-organizing pattern with rotating spiral interfaces at low noises if $\varepsilon = 0$. In this state the three strategies are present with the same frequency and the typical domain size increases with $1/\lambda$. The quantitative features of this behavior remain unchanged for very low values of ε as it is demonstrated in Fig. 2. The quantitative analysis of the average strategy frequencies illustrates the appearance of the paradoxical effect first reported by Tainaka [44] when ε is increased. Fig. 3 shows that only $\bar{\rho}_P$ increases monotonously with the strength ε if $\varepsilon < \varepsilon_{th}$ for fixed K and λ. In other words, the external preference of strategy R is suppressed by the cyclic dominance component and strategy P (the predator of R) will dominate the system behavior at low noises if $\varepsilon < \varepsilon_{th}$ (here $\varepsilon_{th} = 0.0765(5)$). In the opposite case ($\varepsilon > \varepsilon_{th}$) the external preference of Rs overcomes the paradoxical effect and strategy R will dominate the system behavior at low noises as it is clearly illustrated by Fig. 1.

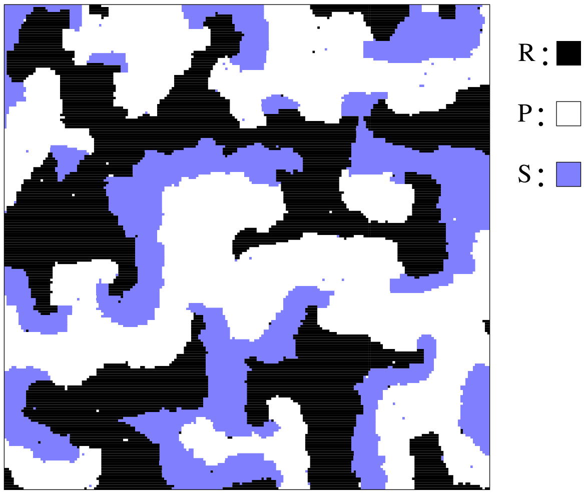

Fig. 2. Rotating spiral interfaces in a typical snapshot of strategy distributions for $K = 0.5$, $\varepsilon = 0.01$ and $\lambda = 0.1$. The strategies R, P, and S are denoted by black, white, and blue pixels on a small part (with 200×200 sites) of the whole lattice.

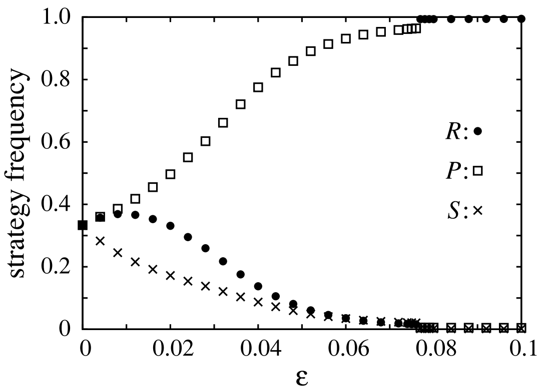

Fig. 3. Strategy frequencies versus ε for $\lambda = 0.1$, $K = 0.7$, and $L = 1000$.

In order to demonstrate what happens if the cyclic component is increased for a fixed ε, in Fig. 4 we show $\bar{\rho}_P$ as a function of λ for three characteristic noise levels. At a low noise ($K = 0.5$) the Potts and cyclic components enforce the formation of a self-organizing domain structure (see Fig. 2) for which the external support of strategy R will increase the frequency of strategy P if $\lambda > \lambda_{th}$ ($\lambda_{th} = 0.117(1)$ for $\varepsilon = 0.1$). In the opposite case ($\lambda < \lambda_{th}$) the system evolves into a state dominated by strategy R if the simulations are started from a random initial state. Here we have to emphasize that in the vicinity of this threshold value ($\lambda \simeq \lambda_{th}$) states dominated by P ($\bar{\rho}_P \simeq 1$) or R ($\bar{\rho}_R \simeq 1$) can be observed for a long time (e.g., $t > 10^6$ MCS if $L = 2000$) at such a low noise level. This phenomenon can cause hysteresis when these variables are slowly tuned at low noises. For higher noises (e.g. at $K = 0.8 < K_c$) the sudden variations in the strategy frequencies are decreased and smoothed out when decreasing the value of λ. Notice that here the effect of ε determines the stationary state dominated by strategy R (i.e., $\bar{\rho}_R \simeq 1$) for sufficiently low values of λ. The third set of MC data in Fig. 4 represents a typical behavior if the high noise level ($K = 1.4 > K_c$) prevents the formation of ordered phases.

The complex effects of noise K on the values of $\bar{\rho}_i$ are demonstrated in Fig. 5 for fixed λ and ε when $\varepsilon < \varepsilon_{th}$. As shown, increasing K results in drastic variations in the average strategy frequencies. Namely, the dominance of strategy P is changed continuously to the dominance of Rs within a narrow range of K. If K is increased then the R dominance is transformed slowly into the random strategy distribution ($\bar{\rho}_i = 1/3$). Here it is worth mentioning that similar phenomena were observed in systems where the imitation type dynamics is extended by the introduction of mutation or spatial dispersal [45]. Recently Nagatani et al. [46] have reported that the paradoxical effect of the cyclic component can be suppressed by sufficiently high mutation rates in biological systems. The significant role of mutation, however, was observed previously for many other three-strategy spatial evolutionary games where all three relevant elementary components are present [47]. The mentioned phenomena are accompanied by variations in the evolution of spatio-temporal patterns.

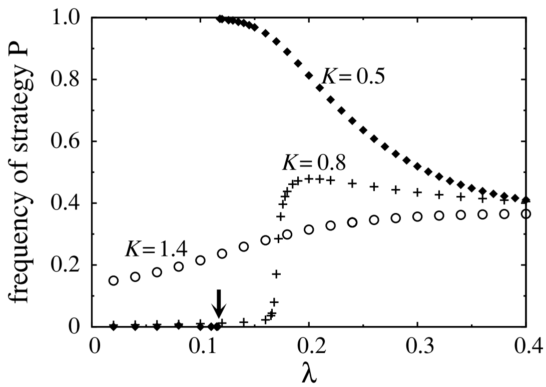

Fig. 4. Frequencies of strategy P versus the strength λ of cyclic component for $\varepsilon = 0.1$, $L = 1000$, and three noise levels indicated ($K = 0.5$, 0.8, and 1.4). The vertical arrow shows the value of λ_{th} for $K = 0.5$.

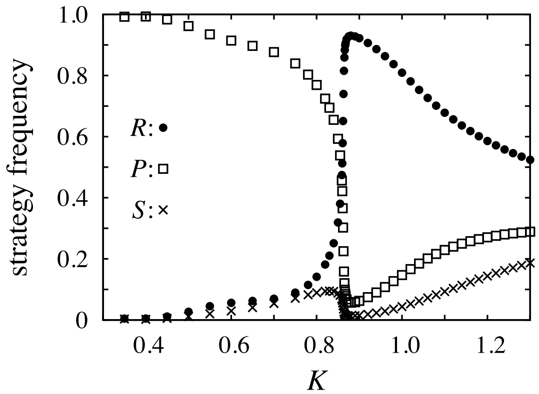

Fig. 5. Strategy frequencies versus K if $\varepsilon = 0.05$, $\lambda = 0.1$, and $L = 2000$.

If the simulation is started from a random initial state for $\varepsilon > \varepsilon_{th}$ at low noises, then at first the visualization of the evolutionary process indicates the formation of small domains of the three strategies. The subsequent domain growing process is governed by the expansion of the domains of strategy R at the expense of others. Due to the randomness in the strategy choices the number of isolated foreign strategies and also the sizes of the emerging small foreign domains increase with K. In fact, in the sea of Rs the domains of Ps can occur with higher probability and expand slowly. On the other hand, S domains can also appear in the islands of Ps and expand faster. Consequently, these consecutive processes maintain the dominance of strategy R. A typical spatial strategy distribution is shown in the right snapshot of Fig. 6. The left snapshot of Fig. 6 illustrates the spatial pattern when $\bar{\rho}_P \simeq 1$. In these cases ($\varepsilon < \varepsilon_{th}$ at low noises) the evolution of the random initial strategy distribution differs from those mentioned above because here the territories of strategy S are quickly occupied by players of strategy R. In the absence of strategy S the territory of strategy P is not invaded. Consequently, it expands slowly at the expense of strategy R, and finally the system evolves into a state dominated by strategy P at low noises. The corresponding paradoxical effects are also reported in many models of three-strategy evolutionary games [2] based on imitation.

The homogeneous distribution of strategy P (or R) is unstable against the extension of S (or P) islands emerging via a nucleation process. For both cases the domain growths are stopped and reversed after the emergence of the third type of domains. This evolutionary process is demonstrated in Fig. 7 where the simulation is started from a prepared initial state in which a circular domain of S is inserted into the sea of Ps. In this example the radius (10 lattice units) of the circular domain of S was sufficiently large to ensure its expansion.

According to the theory of nucleation a sufficiently large domain of the preferred phase can emerge via a long series of stochastic events [48–50] and this nucleus initiates the evolution towards the thermodynamically preferred phase. In the present system, however, the emergence of R domains (via a similar nucleation process) suppresses the growth of S domains and finally the system evolves into a phase dominated by P as illustrated by the time-dependent strategy frequencies in Fig. 8. Here short-time fluctuations are suppressed by averaging over windows of 20 MCS.

In short, the birth and death of these bursts are initiated by two (or more) consecutive nucleation processes and the average size of the "avalanches" is affected by the K-dependence of invasion velocities and nucleation rates. Previous theoretical analyses indicate that the probabilities of the formations of these nuclei decrease drastically if the noise level

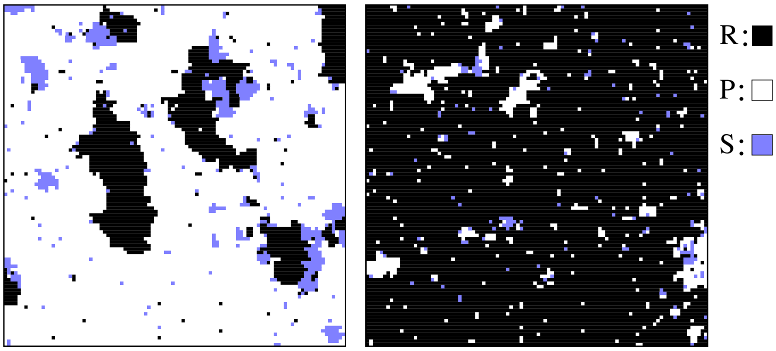

Fig. 6. Typical strategy distributions obtained for $K = 0.7$ (left) and 0.9 (right) if $\varepsilon = 0.05$ and $\lambda = 0.1$. These snapshots show small parts (with 100×100 sites) of the whole lattice.

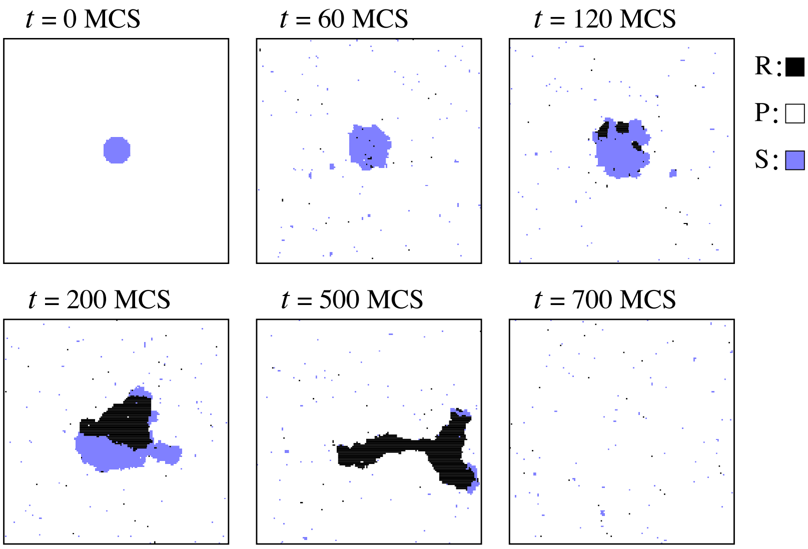

Fig. 7. Series of snapshots illustrating a typical burst process for $\varepsilon = 0.05$, $\lambda = 0.1$, and $K = 0.6$ if initially ($t = 0$ MCS) a small circular island of S is substituted into the homogeneous P phase.

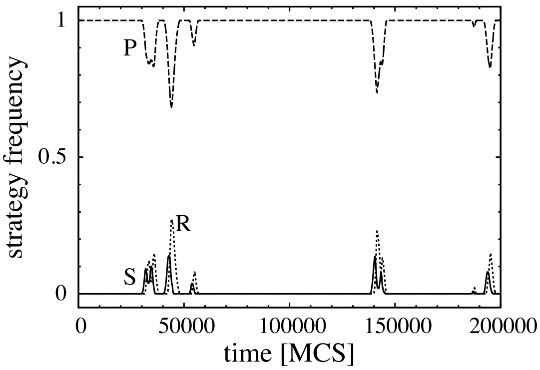

Fig. 8. Quantification of bursts in the time dependence of strategy frequencies for $\varepsilon = 0.05$, $\lambda = 0.1$, $K = 0.42$, and $L = 1000$. The dotted, dashed and solid lines refer to strategies R, P and S.

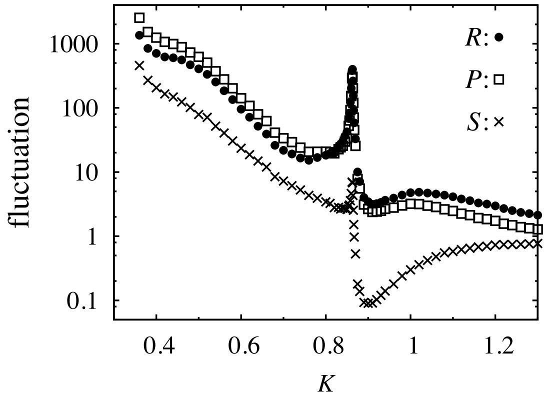

Fig. 9. Noise dependence of the fluctuations of strategy frequencies for $\varepsilon = 0.05$ and $\lambda = 0.1$ ($L = 2000$).

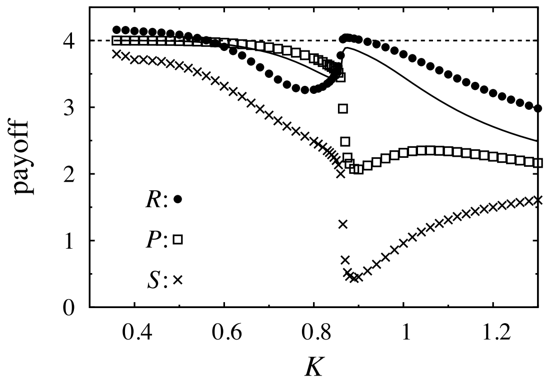

Fig. 10. Strategy-dependent average payoffs $\bar{p}_i$ as a function of noise for $\varepsilon = 0.05$ and $\lambda = 0.1$ ($L = 2000$). The solid line shows the average value over the whole society while the thin dashed line serves as a reference indicating the average payoff in the homogeneous P state.

is decreased. In these cases the bursts occur more and more sparsely while their extent (both in time and space) may become larger and larger. This phenomenon is accompanied by a relevant increase of fluctuations (χ_i) in the strategy frequencies (see Fig. 9) when decreasing K.

Notice the simultaneous peaks in χ_R and χ_P at the K value where the dominating strategies are exchanged. These peaks are related to the sharp variations in $\bar{\rho}_R$ and $\bar{\rho}_P$.

Here it is worth mentioning the technical difficulties reducing the accuracy of the MC data at low noises. Despite the large system size and long sampling time for the lowest noise levels we could observe only a small number of bursts during the simulations. For example, at $K = 0.36$ the MC data are extracted from an evolutionary process in which only 51 individual bursts could be identified for $L = 2000$ and $t_s \simeq 7 \cdot 10^7$ MCS.

Similar bursts were observed in many social and biological models [51,52]. The systematic analysis of the general features of these bursts [53–56], however, goes beyond the scope of the present work. We emphasize, however, that similar phenomena may occur in many other three-state lattice systems where the mean-field approximation predicts asymptotic evolution towards a fixed point which is unstable [2,4].

The quantification of the average payoffs $\bar{p}_i$ has indicated another curious feature (see Fig. 10) related to the bursts. In the limit $K \to 0$ the average payoff in the whole system goes to 4 while the solitary R and S players receive significantly lower payoffs. According to the MC data, however, the average payoff of the R players may be higher because most of these players reside in sufficiently large domains providing the highest possible income for them. Fig. 10 shows two ranges of the noise parameter K in which $\bar{p}_R > 4$. At low noises the bursts ensure the highest average strategy specific income for the players of strategy R. At the same time the average payoff $\bar{p}$ remains below 4 because the players receive low payoffs along the interfaces. This phenomenon represents a type of social dilemmas that forces selfish players to choose a strategy which is not optimal for them.

For the logit rule (4) the frequency of wrong decisions increases with K. Thus, one can naively expect that the average payoff $\bar{p}$ decreases if K is increased. By contrast, Fig. 10 shows that $\bar{p}$ has a local maximum when the dominance of R becomes sufficiently high. The further increase of K, however, drives the system towards the random strategy distribution when the high contributions of the Potts component are suppressed.

4. Summary

We have studied a three-strategy (denoted as R, P, and S) spatial evolutionary ordinal potential game with three parameters characterizing the strength of external effect, cyclic dominance, and noise level. The dominant part of pair interaction is described by a Potts component ensuring the existence of three pure Nash equilibria (for the two-player games) and the stability of the corresponding three homogeneous strategy distributions on a square lattice at low noises. We have focused our efforts on the phenomena caused by the interplay between the external support (favoring strategy R) and cyclic dominance represented by traditional rock–paper–scissors game.

At low noise levels the Monte Carlo simulations have indicated the dominance of strategy R if the strength of cyclic component is smaller than a threshold value. In these cases the strategy frequencies exhibit a noise dependence similar to one described by the Potts model in the presence of an external field although the main characteristics of thermodynamical behavior (Gibbs distribution, detailed balance, etc.) are broken.

For stronger cyclic component the strategy P (which is the predator of the externally supported R) will dominate the system behavior at low noises in agreement with the prediction of the paradoxical effect reported by Tainaka [44]. The homogeneous distribution of strategy P, however, is unstable because strategy S can occasionally form growing domains (bursts) via the traditional nucleation process. In parallel with this process strategy R can also occur via a similar nucleation process in the domains of S. The territories of Rs are invaded by P players and finally these bursts are eliminated. In this system the evolution of bursts (avalanches) is controlled by the corresponding nucleation rates and invasion velocities, and it causes huge fluctuations in the strategy frequencies. Our MC data indicate rarer, larger, and longer bursts with wider probability distributions if the noise level is decreased. Due to the formation of large domains these bursts provide an enhanced average payoff for the R players and their elimination by P players can be interpreted as a "tragedy of the commons".

The numerical analysis of the noise-dependence has shown that the dominance of strategy P ($\bar{\rho}_P > 0.8$) develops sharply into the dominance of R players ($\bar{\rho}_R > 0.8$) in a narrow range of K where the average payoff of the whole community reaches a local maximum. In this R-dominated phase the visualization of the spatial distribution of strategies shows growing and invaded domains of Ps. In agreement with expectations the further increase of noise drives the system towards the random strategy distribution.

In this system all three homogeneous phases can be unstable against the invasions of the islands of a suitable (predator) strategy which are created by nucleation processes. Due to the cyclic component and the logit rule, the predator of the growing domain will also occur sooner or later and the final compositions of the self-organizing patterns are determined by the interplay among these processes which can result in fundamentally different behaviors and spatiotemporal patterns as illustrated by our result.

The exhaustive analysis of the transitions between these phases and the exploration of the general features of the spatio-temporal patterns and mechanisms go beyond the scope of the present work. The above results clearly indicate that further systematic studies and the development of new approaches are necessary to fully explore how tuning the noise and the ratio of the strengths of elementary interaction coefficients affect the general features of this simple evolutionary game.

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