Phase transition classes in triplet and quadruplet reaction-diffusion models

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Phase transitions of reaction-diffusion systems with site occupation restriction and with particle creation that requires $n=3,4$ parents, whereas explicit diffusion of single particles (A) is present are investigated in low dimensions by the mean-field approximation and simulations. The mean-field approximation of general $nA \to (n+k)A$, $mA \to (m-l)A$ type of lattice models is solved and a different kind of critical behavior is pointed out. In $d=2$ dimensions, the $3A\to 4A$, $3A\to 2A$ model exhibits a continuous mean-field type of phase transition, that implies $d_c<2$ upper critical dimension. For this model in $d=1$ extensive simulations support a mean-field type of phase transition with logarithmic corrections unlike the recent study of Park *et al.* [Phys. Rev E **66**, 025101 (2002)]. On the other hand, the $4A\to 5A$, $4A\to 3A$ quadruplet model exhibits a mean-field type of phase transition with logarithmic corrections in $d=2$, while quadruplet models in one-dimensional show robust, nontrivial transitions suggesting $d_c=2$. Furthermore, I show that a parity conserving model $3A\to 5A$, $2A\to \emptyset$ in $d=1$ has a continuous phase transition with different kinds of exponents. These results are in contradiction with the recently suggested implications of a phenomenological, multiplicative noise Langevin equation approach and with the simulations on suppressed bosonic systems by Kockelkoren and Chaté [Phys. Rev. Lett. **90**, 125701 (2003)].

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I. INTRODUCTION

Phase transitions in genuine nonequilibrium systems have been investigated often among reaction-diffusion (RD) type of models exhibiting absorbing states [1–3]. In many cases, mapping to surface growth, spin systems, or stochastic cellular automata can be done. The classification of universality classes of second-order transitions is still one of the most important uncompleted tasks. One hopes that symmetries and spatial dimensions are the most significant ingredients as in equilibrium cases, however, it turned out that in many cases there is a shortage of such factors to explain the emerging universality classes. An important example was being investigated during the past two years which emerges at phase transitions of binary production systems [4–16]: the diffusive pair contact process (PCPD). In these systems, particle production competes with pair annihilation and single particle diffusion. If the production wins, steady states with finite particle density appear in (site restricted) models with hard-core repulsion, while in unrestricted (bosonic) models the density diverges. By lowering the production per annihilation rate, a doublet of absorbing states without symmetries emerges. One of such states is completely empty, the other possesses a single wandering particle. In case of site restricted systems, the transition to absorbing states is continuous.

Although the nature of this transition has not completely been settled numerically and by field theory yet, another novel class appearing in triplet production systems was proposed very recently [17,18]: the diffusive triplet contact process (TCPD). This reaction-diffusion model differs from the PCPD model in such a way that for a new particle generation at least three particles have to meet. It is important to note that these models do not break the directed percolation (DP) hypothesis [19,20]—according to which, *in one component systems exhibiting continuous phase transitions to single absorbing state (without extra symmetry and inhomogeneity or disorder), short ranged interactions can generate DP class transition only*—because they exhibit multiple absorbing states that are not frozen, lonely particle(s) may diffuse in them.

A phenomenologically introduced Langevin equation that exhibits real, multiplicative noise was suggested [17] to describe the critical behavior of reaction-diffusion models of types

$$nA\to(n+1)A, \quad nA\to jA, \tag{1}$$

(with $j<n$, number of interacting particles) in the form

$$\partial_t \rho(x,t) = a\rho(x,t)^n - \rho(x,t)^{n+1} + D\nabla^2\rho(x,t) + \zeta(x,t), \tag{2}$$

with noise correlations

$$\langle \zeta(x,t)\zeta(x',t')\rangle = \Gamma \rho^{\mu}\delta^d(x-x')\,\delta(t-t'). \tag{3}$$

The classification of universality classes of nonequilibrium systems by the exponent μ of a multiplicative noise in the Langevin equation was suggested some time ago by Grinstein *et al.* [21]. However, it turned out that there may not be corresponding particle systems to real multiplicative noise cases [4] and an imaginary part appears as well if one derives the Langevin equation of a RD system starting from the master equation in a proper way. This observation led Howard and Täuber to investigate systems with complex noise appearing in binary production models. Unfortunately, the cases with and without occupation number restriction turned out to be different in $d=1$, although in $d=2$ this difference was found to disappear at and below criticality [13].

By rescaling Eq. (2), one can get the corresponding mean-field critical exponents

$$\beta^{MF}=1, \quad \nu_{\perp}^{MF}=n/2, \quad \nu_{||}^{MF}=n. \tag{4}$$

The authors of Ref. [17] expect that the noise exponent should be in the range

$$1 \leqslant \mu \leqslant n, \tag{5}$$

hence by simple power counting, the upper critical dimension should be

$$d_c = 2 + \frac{4-2\mu}{n}. \tag{6}$$

This implies that for a triplet processes, $4/3 \leqslant d_c \leqslant 8/3$ and for a quadruplet $(n=4)$ processes: $1 \leqslant d_c \leqslant 5/2$.

Very recently, Kockelkoren and Chaté introduced stochastic cellular automata (SCA) versions of general $nA \to (n+k)A$, $mA \to (m-l)A$ type of models [18], where multiple particle creation on a given site is suppressed by an exponentially decreasing creation probability $p^{N/2}$ of the particle number. They claim that their simulation results in one-dimension are in agreement with the fully occupation number restriction counterparts and set up a general table of universality classes, where as the function of n and m only four non-mean-field classes exist, namely, the directed percolation class [19,20] the parity conserving class [22], the PCPD, and the TCPD classes.

In any case, the heuristic Langevin equation with real noise assumption for RD models [17,18] should be proven for $n>1$. Furthermore, in low dimensions topological constraints may cause a different critical behavior with and without occupation number restriction [23]. Note that in case of binary production models, it had not been clear at all if the $d_c=2$, prediction of the bosonic field theory had also been true for site restricted systems until the numerical confirmation of Ref. [13]. In this paper, I show simulation results for lattice models with restricted site occupancy in $d=1,2$ with the aim of locating the upper critical dimensions and checking claims of Refs. [17,18] about possible new universality classes.

II. MEAN-FIELD CONSIDERATIONS

In this Sec. I, discuss the mean-field equation that can be set up for site restricted lattice models with general microscopic processes of the form

$$nA \xrightarrow{\sigma} (n+k)A, \quad mA \xrightarrow{\lambda} (m-l)A, \tag{7}$$

with $n>1$, $m>1$, $k>0$, $l>0$, and $m-l \geqslant 0$. Note that this formulation is different from that of Eq. (2), that is suggested for coarse grained, continuous bosonic description of these reaction-diffusion systems. In this case the diffusion drops out and one can neglect the noise, hence the competition of creation (with probability σ) and annihilation or coagulation (parametrized with probability $\lambda = 1-\sigma$) is left behind:

$$\frac{\partial \rho}{\partial t} = ak\sigma\rho^n(1-\rho)^k - al(1-\sigma)\rho^m, \tag{8}$$

where ρ denotes the site occupancy probability and a is a dimension dependent coordination number. Each empty site has a probability $(1-\rho)$ in the mean-field approximation, hence the need for k empty sites at a creation brings in a $(1-\rho)^k$ probability factor. By expanding $(1-\rho)^k$ and keeping the lowest-order contribution, one can see that for site restricted lattice systems a ρ^{n+1}th order term appears automatically with negative coefficient that regulates Eq. (8). The steady state solution can be found analytically in many cases and may result in different, continuous or discontinuous phase transitions. Here I split the discussion of the solutions to three parts: (a) $n=m$, (b) $n>m$, and (c) $n<m$. In the inactive phases, one expects a dynamical behavior described by the $mA \to \emptyset$ process, for which $\rho \propto t^{1/(m-1)}$ is known [22].

A. The $n=m$ symmetric case

The steady state solution in this case can be obtained by solving

$$k\sigma(1-\rho)^k = l(1-\sigma), \tag{9}$$

where the trivial $(\rho=0)$ solution has been factored out. For the active phase, one gets

$$\rho = 1 - \left[\frac{l}{k}\,\frac{1-\sigma}{\sigma}\right]^{1/k}, \tag{10}$$

which vanishes at $\sigma_c = l/(k+l)$ with the leading order singularity

$$\rho \propto |\sigma - \sigma_c|^{\beta^{MF}}, \tag{11}$$

and order parameter exponent $\beta^{MF}=1$. At the critical point, the time dependent behavior is described by

$$\frac{\partial \rho}{\partial t} = -2ak^2\rho^{n+1} + O(\rho^{n+2}), \tag{12}$$

that gives a leading order power-law solution

$$\rho \propto t^{-1/n}, \tag{13}$$

hence

$$\alpha_{MF} = \beta^{MF}/\nu_{||}^{MF} = 1/n.$$

This was obtained from bosonic, coarse grained formulation in Ref. [17] too.

B. The $n>m$ case

In this case besides the $\rho=0$ absorbing state solution we can get an active state if

$$k\sigma\rho^{n-m}(1-\rho)^k = l(1-\sigma) \tag{14}$$

is satisfied. Both sides are linear functions of σ, such that for $\sigma \to 0$ only the $\rho=0$ is a solution. The left-hand side is a convex function of ρ (from above) with zeros at $\rho=0$ and $\rho=1$.

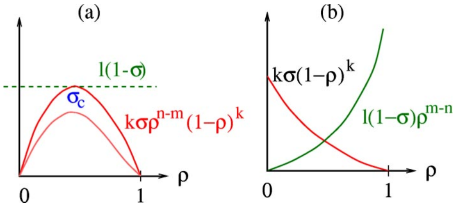

FIG. 1. Steady state mean-field solution for (a) $n>m$ and (b) $n<m$ cases.

Therefore by increasing σ from zero, the left-hand side meets the right-hand side at σ_c, $\rho_c>0$ [See Fig. 1(a)]. If this solution is stable, a first-order transition takes place in the system. Note that in higher order cluster mean-field solutions, where the diffusion can play a role, the transition may turn into a continuous one [24–26]; therefore it is important to check the type of transition for $d \geqslant d_c$. In Sec. III C, I shall confirm the first orderedness of such transitions for two models in two-dimension.

C. The $n<m$ case

By factoring out the trivial $\rho=0$ solution, we are faced with the general condition for a steady state

$$k\sigma(1-\rho)^k = l(1-\sigma)\rho^{m-n}. \tag{15}$$

One can easily check that in this case, the critical point is at $\sigma_c=0$ [see Fig. 1(b)] and here the density decays with $\alpha^{MF}=1/(m-1)$ as in case of the $n=1$ branching and $m=l$ annihilating models, showed by Cardy and Täuber [22] branching and k-annihilating random walk (BkARW classes). However, the steady state solution for $n>1$ gives different β exponents than those of BkARW classes, namely, $\beta^{MF}=1/(m-n)$. This implies a different kind of critical behavior in low dimensions which could be a subject of further investigation.

III. SIMULATIONS IN TWO DIMENSIONS

Two-dimensional simulations were performed on $L=400-1000$ linear sized lattices with periodic boundary conditions. One Monte Carlo step (MCS)—corresponding to $dt=1/P$ (where P is the number of particles)—is built up from the following processes. A particle and a number $x \in (0,1)$ are selected randomly; if $x<D$, a site exchange is attempted with one of the randomly selected empty nearest neighbors (nn); if $x \geqslant D$, k number of new particles are created with probability $(1-p)$ at randomly selected empty nn sites provided the number of nn particles was greater than or equal to n; or if $x \geqslant D$, l number of particles are removed with probability p (taking into account the $m-l \geqslant 0$ condition as well). The simulations were started from fully occupied lattices, and the particle density decay was measured up to 10^6–10^7 MCS.

A. The $3A \to 4A$, $3A \to 2A$ symmetric triplet model

First, I checked the dynamic behavior in the inactive phase for $D=0.5$ diffusion rate. At $p=0.9$, one can see the appearance of the mean-field behavior $\rho(t) \propto t^{-1/2}$ following 2×10^6 MCS. By decreasing p, this scaling sets in later times. As Fig. 2 shows for $L=1000$ systems with $t_{max}=10^7$ MCS curves with $p \leqslant 0.4965$ veer up—corresponding to the active phase—while curves with $p \geqslant 0.497$ veer down—corresponding to the absorbing state. From the $\rho(t)$ data, I determined the effective exponents (the local slopes) defined as

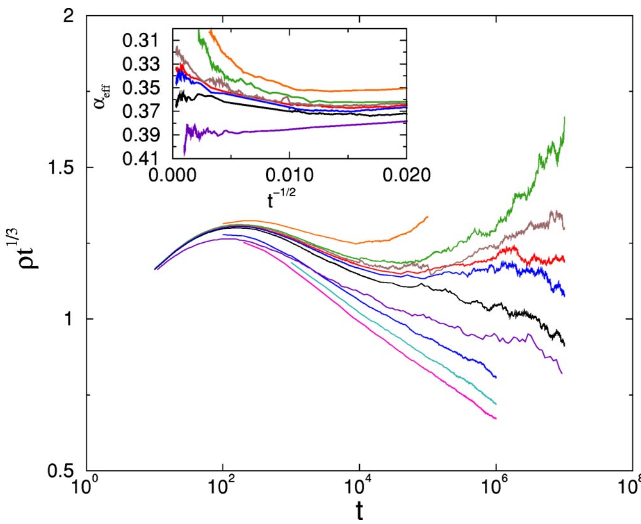

FIG. 2. Density times $t^{1/3}$ in the two-dimensional $3A \to 4A$, $3A \to 2A$ model for $D=0.5$, and $p=0.496$, 0.4965, 0.4967, 0.4968, 0.497, 0.498, 0.4985, and 0.499 (top to bottom curves). The inset shows the corresponding local slopes.

$$\alpha_{eff}(t) = \frac{-\ln[\rho(t)/\rho(t/m)]}{\ln(m)} \tag{16}$$

(where I used $m=4$). The critical point is estimated at $p=0.4967(2)$ with $\alpha=0.33(1)$ (for local slopes, see inset of Fig. 2). This value agrees with the mean-field value $\alpha^{MF}=1/3$.

Density decays for several p's in the active phase $(0.003<\epsilon=|p_c-p|<0.3)$ were followed on logarithmic time scales, and averaging was done over ~ 100 independent runs in a time window, which exceeds the level-off time by a decade. The steady state density in the active phase at a critical phase transition is expected to scale as

$$\rho(\infty,p) \propto |p-p_c|^{\beta}. \tag{17}$$

Using the local slopes method, one can get a precise estimate for β as well as for the corrections to scaling

$$\beta_{eff}(\epsilon_i) = \frac{\ln \rho(\infty,\epsilon_i) - \ln \rho(\infty,\epsilon_{i-1})}{\ln(\epsilon_i) - \ln(\epsilon_{i-1})}, \tag{18}$$

where I used the p_c value determined before. One can see in Fig. 3 that the effective exponent for $\epsilon>0.005$ exhibits a correction to scaling (inclined line) and tends to $\lim_{\epsilon\to 0}\beta=1.0(1)$, which agrees with the mean-field value again. By neither the α nor the β exponent, one can observe logarithmic corrections suggesting $d_c<2$.

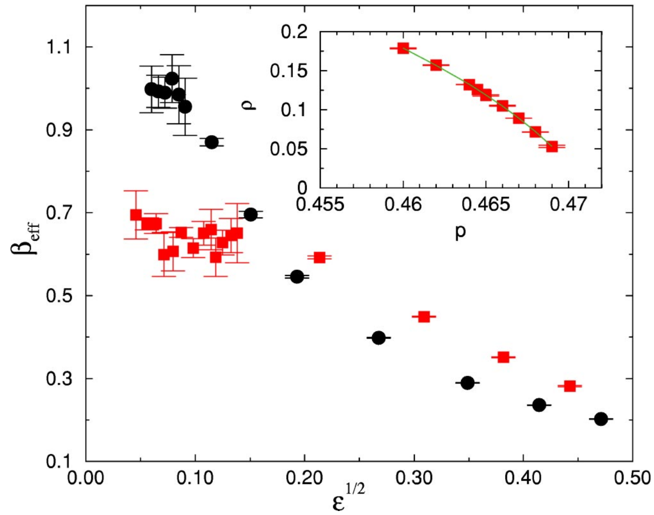

FIG. 3. Effective order parameter exponent results in two dimensions. Bullets correspond to the $3A\to4A$, $3A\to2A$ model and squares to the $4A\to5A$, $4A\to3A$ model at $D=0.5$. The inset shows the logarithmic fitting for the $4A\to5A$, $4A\to3A$ model.

The density decay simulations were repeated at $D=0.2$, where the critical point was found at $p_c=0.4795(1)$ with a mean-field class α exponent again.

B. The $4A\to5A$, $4A\to3A$ symmetric quadruplet model

Here simulations are much slower than in case of the triplet model, hence systems with linear size $L=400$ could be investigated. First, I checked the dynamic behavior in the inactive phase for $D=0.5$. At $p=0.9$, a mean-field type of decay $\rho(t)\propto t^{-1/3}$ can be observed following 10^6 MCS. As one can see in Fig. 4, for $p<0.4705$ the density decay curves veer up, while for $p\geqslant0.471$ these curves veer down. The estimated critical point is $p_c\simeq0.4705(2)$. The effective exponent at p_c extrapolates to $\alpha\simeq0.22(1)$. As one can see in this graph, the separatrix (critical) curve exhibits a linear shape on the $\rho(t)t^{1/4}-\ln(t)$ scale suggesting logarithmic corrections to scaling. Similarly, the effective exponents of β seem to extrapolate to $\beta\simeq0.63(5)$ (Fig. 3), that is very far from the mean-field value $\beta^{MF}=1$. To check the possibility that a logarithmic correction can result in the mean-field exponents, the fitting with the lowest-order correction

$$\rho(\infty,p)=\{(p_c-p)[a+b\ \ln(p_c-p)]\}^{\beta} \qquad (19)$$

has been applied for the steady state $\rho(\infty,p)$ data. I used the nonlinear fitting of the XMGR graphical package with a relative error in the sum of squares with at most 0.0001. This resulted in $\beta=1.01$ at $p=0.471$ ($a=-10.8$, and $b=-6.05$) (see inset of Fig. 3). This result in agreement with the dynamical scaling conclusion may support that the upper critical dimension for quadruplet models is $d_c=2$. To get more solid results further, very extensive simulations would be necessary that is beyond the scope of this study. In any case, the clear mean-field behavior cannot be concluded.

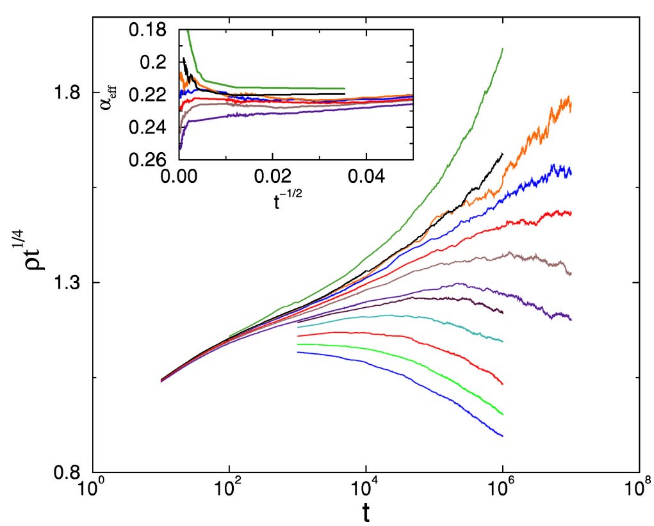

FIG. 4. Density times $t^{1/4}$ in the two-dimensional $4A\to5A$, $4A\to3A$ model for $D=0.5$ and $p=0.469$, 0.47, 0.4792, 0.4705, 0.471, 0.4715, 0.4725, 0.473, 0.474, 0.476, 0.478, and 0.48 (top to bottom curves). The inset shows the corresponding local slopes.

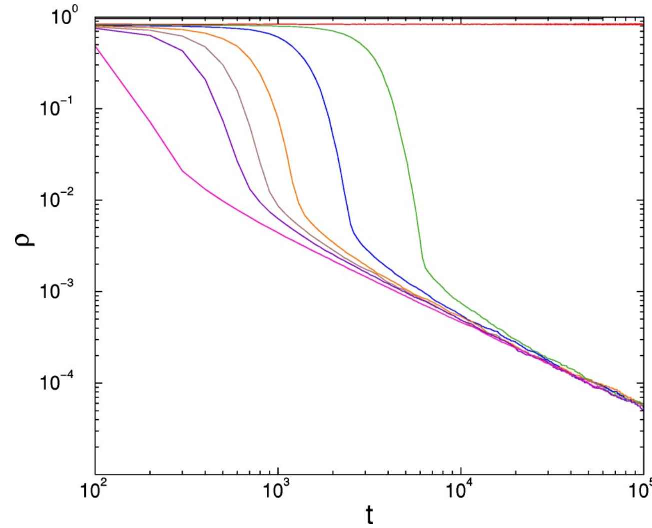

FIG. 5. Density decay in the two-dimensional $4A\to5A$, $2A\to\emptyset$ model for $D=0.5$ and $p=0.05$, 0.119, 0.121, 0.122, 0.123, 0.124, 0.125, and 0.13 (top to bottom curves) with system sizes $L=400$.

C. $3A\to5A$, $2A\to\emptyset$ and $4A\to5A$, $2A\to\emptyset$ hybrid models

One can find two regions in the density decay behavior by varying p in both the models. For $p<p_c$ steady state values are reached quickly, while for $p>p_c$ a rapid (faster than power law) initial density decay crosses over to $\rho\propto t^{-1}$. This is in agreement with the mean-field behavior of the $2A\to\emptyset$ process in one dimension [27], dominating in the inactive phase. For the $4A\to5A$, $2A\to\emptyset$ quadruplet production model, this threshold is at $p_c=0.119(1)$ (see Fig. 5) where an abrupt jump is observable from $\rho(\infty)=0.833$ to $\rho(\infty)=0$. In case of the $3A\to5A$, $2A\to\emptyset$ triplet production model, the threshold is at $p_c=0.220(1)$ with a jump from $\rho(\infty)=0.45$ to zero.

In neither cases do we see dynamical scaling at the transition. These results are in agreement with the first-order transition of the mean-field approximations, given in Sec. II B for $n>m$.

IV. SIMULATIONS IN ONE DIMENSION

The simulations in one dimension were carried out on $L=10^5$ sized systems with periodic boundary conditions. The initial states were again fully occupied lattices, and the density of particles is followed up to 10^6-10^7 MCS. An elementary MCS consists of the following processes: (a) $A\emptyset\leftrightarrow\emptyset A$ with probability D, (b) $mA\to(m-l)A$ with probability $p(1-D)$, (c) $nA\to(n+k)A$ with probability $(1-p)(1-D)$, such that the reactions were allowed on the left or right side of the selected particle strings.

A. $3A\to4A$, $3A\to2A$ and $3A\to6A$, $3A\to\emptyset$ symmetric triplet models

The $3A\to4A$, $3A\to2A$ site restricted model in one dimension was simulated by Park *et al.* [17] for small systems up to 10^6 MCS. They concluded to find a kind of phase transition with the order parameter exponents $\alpha=0.32(1)$ and $\beta=0.78(3)$. For the restricted bosonic version of this model, large scale simulations gave $\alpha=0.27(1)$ and $\beta=0.90(5)$ [18]. Note that since reactive and diffusive sectors arise in this model like in PCPD class model, diffusion dependence or corrections to scaling may hamper to see real asymptotic behavior [7,39,40]. Here, I show extended simulation results for the strictly site restricted lattice model model with $t_{max}=10^7$ MCS at diffusion rate $D=0.1$. At the critical point the $\alpha_{eff}(t)$ curve exhibits a straight line shape for $t\to\infty$, while in sub-critical (supercritical) cases $\alpha_{eff}(t)$ curves veer down (up), respectively. As one can see in Fig. 6, following a long relaxation, $p\leqslant0.3032$ curves veer up, while $p\geqslant0.3035$ curves veer down in the $t\to\infty$ limit. From this, one can estimate $p=p_c\simeq0.3033(1)$ with $\alpha=0.33(1)$ in agreement with the results of Ref. [17].

By analyzing supercritical steady state densities with the local slopes method, one can read off: $\beta_{eff}\to\beta\simeq0.95(5)$ (see Fig. 7), which is higher than the results of Refs. [17] and [18].

However, one should be careful and check diffusion dependence and corrections to scaling, especially, because these critical exponent estimates are quite close to the mean-field values ($\alpha^{MF}=1/3$ and $\beta^{MF}=1$) and as it was shown in Sec. III, $d_c<2$. Since the $p=0.303$ and $p=0.3035$ $\rho(t)$ curves show clear curvature for large times, the $0.303<p_c<0.3035$ conditions seem to be inevitable.

I tried to fit the steady state data in the $0.303\leqslant p\leqslant0.3035$ region by the logarithmic correction form (19) and obtained $\beta=1.07(10)$ at $p_c=0.3032$ that agrees with the mean-field value and implies $d_c=1$.

Just considering mean-field results, according to which k does not play a role for $n=m$ models, one may expect that the $3A\to6A$, $3A\to\emptyset$ triplet creation model exhibits the same kind of transition as the $3A\to4A$, $3A\to2A$ model. Indeed, Kockelkoren and Chaté's simulations show this [18]. However, doing lattice simulations with site restrictions, it turned out that the $3A\to6A$ creation was so effective that it

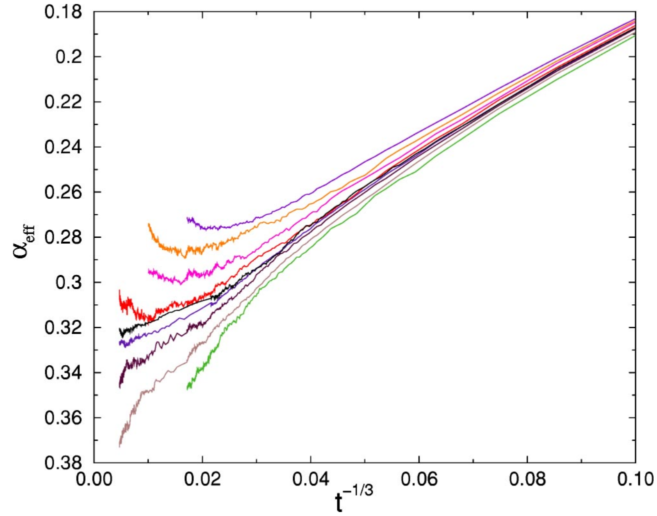

FIG. 6. Local slopes of the density decay in the one-dimensional $3A\to4A$, $3A\to4A$ model at $D=0.1$. Different curves correspond to $p=0.3$, 0.301, 0.3015, 0.302 0.3025, 0.3027, 0.303, 0.3035, 0.304, and 0.3045 (from top to bottom).

shifted the transition to the zero production limit ($p=1$) where the $3A\to\emptyset$ process in one dimension is known to decay as $\rho\propto[\ln(t)/t]^{1/2}$ [22]. Off-critical simulations showed that $\beta=0.33(1)$, meaning that this transition belongs to the BkARW mean-field class. On the other hand there may be other realizations of this model, where the transition reported by Kockelkoren and Chaté is accessible.

B. The $3A\to5A$, $2A\to\emptyset$ model

It has been established that in $n=1$, $m=l=2$, and even k—so called even number of offspringed branching and annihilating models (BARWe)–the parity conserving (PC)

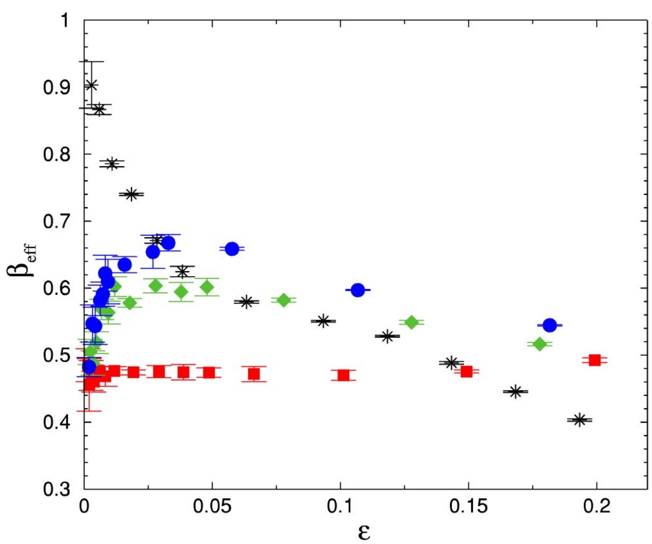

FIG. 7. Effective order parameter exponent results in one dimension. Stars correspond to $3A\to4A$, $3A\to2A$ model; bullets to $4A\to5A$, $2A\to\emptyset$ model at $D=0.2$; squares to $4A\to5A$, $2A\to\emptyset$ model at $D=0.8$; and diamonds to $4A\to5A$, and $4A\to\emptyset$ model at $D=0.3$.

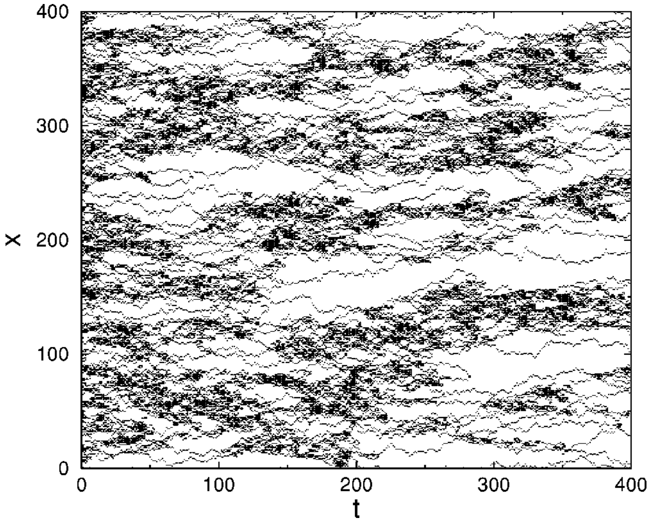

FIG. 8. Spatiotemporal evolution of the critical, (1+1)-D diffusive $3A\rightarrow5A$, $2A\rightarrow\emptyset$ model ($D=0.8$).

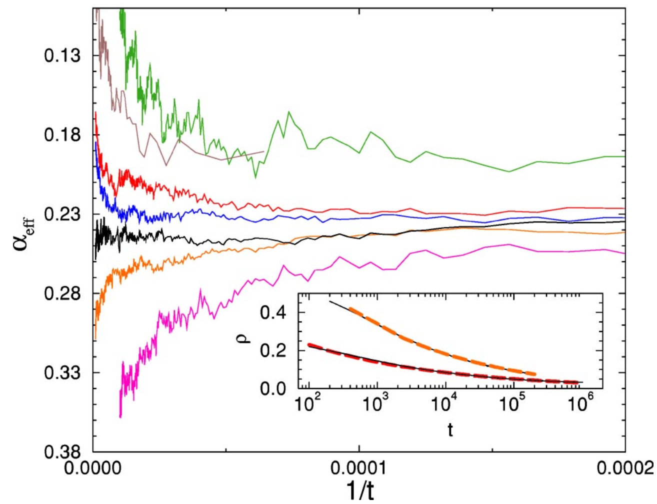

FIG. 9. Local slopes of the density decay in the one-dimensional $3A\rightarrow5A$, $2A\rightarrow\emptyset$ model for $D=0.8$. Different curves correspond to $p=0.46$, 0.461, 0.462, 0.4625, 0.4627, 0.463, 0.46325, 0.464, and 0.465 (from top to bottom).

class continuous phase transition emerges [22,29,30]. This class has also been observed in models exhibiting Z_2 symmetric absorbing states, where the domain walls separating ordered phases follow BARWe dynamics [31–34,37]. This class was originally called parity conserving, owing to the conservation law that made it different from the robust DP class. However, it turned out that in Z_2 symmetric models this conservation is not enough [35–38]. Furthermore, in binary spreading models, this conservation was found to be irrelevant [10,13]. Therefore, it is still an open question whether parity conservation is relevant in other models than in BARW types.

I investigated the phase transition of the triplet production $3A\rightarrow5A$, $2A\rightarrow\emptyset$ model (with explicit particle diffusion) possessing parity conservation. As I showed in Sec. III C, in two dimensions this system exhibits a first-order transition in agreement with the mean-field results. This first-order mean-field behavior does not give a direct hint on the type of phase transition in one dimension. Kockelkoren and Chaté's simulations on the one-dimensional, suppressed bosonic cellular automaton version of this model show simple DP class density decay [18]. However, if we consider the space-time evolution, we see very non-DP like spatiotemporal pattern (see Fig. 8). This pattern resembles much more to those of the PCPD class models, where compact domains separated by clouds of lonely wandering particles occur. Of course, such qualitative judgment on the universal behavior is not enough, but has been found to be quite successful in case of binary production systems [8,11].

The density decay simulations at $D=0.8$ and $D=0.2$ have been analyzed by the local slopes method, see Fig. 9. At $D=0.8$, the critical point is estimated at $p_c^H=0.4629(3)$ and the corresponding effective exponent tends to $\alpha^H=0.24(1)$. At $D=0.2$, the critical point is at $p_c^L=0.2240(3)$ and the local exponent seems to extrapolate to $\alpha^L=0.28(1)$. Such small difference between the high and low D results can also be observed by analyzing the steady state results: $\beta^H=0.43(3)$ versus $\beta^L=0.63(3)$. These exponent estimates are far from the (1+1)-D DP values ($\alpha=0.159\,464(6)$, $\beta=0.276\,486(8)$ [41]), hence the claim of Kockelkoren and Chaté for the critical behavior of $n>m$ models is questionable.

On the other hand, the diffusion dependence of the critical exponents is a challenge and has been observed in the binary production PCPD model [7]. In Ref. [40], it was shown that assuming logarithmic corrections to scaling—that is quite common in the one-dimensional models—a single universality class can be supported numerically. Therefore, here again I have investigated the possibility of the collapse of the high and low D exponents. Assuming the same kind of logarithmic correction forms as in Ref. [40],

$$[[a+b\,\ln(t)]/t]^{\alpha}. \tag{20}$$

I have found a consistent set of exponents both for $D=0.2$ and $D=0.8$ (see Table I and inset of Fig. 9). For the data analysis, I used the nonlinear fitting of the program XMGR package, with a relative error in the sum of squares with at most 0.001,

$$\alpha=0.22(1), \quad \beta=0.60(1), \tag{21}$$

with the critical thresholds: $p_c^H=0.4627(1)$ and $p_c^L=0.2240(1)$. These exponents suggest a distinct universality class from the known ones [3].

TABLE I. Logarithmic fitting results by the form (20) for the one-dimensional $3A\rightarrow5A$, $2A\rightarrow\emptyset$ model.

D	p_c	a	b	α
0.2	0.4627	0.115	3.5×10^{-5}	0.217
0.8	0.22405	14.12	-1.001	0.224

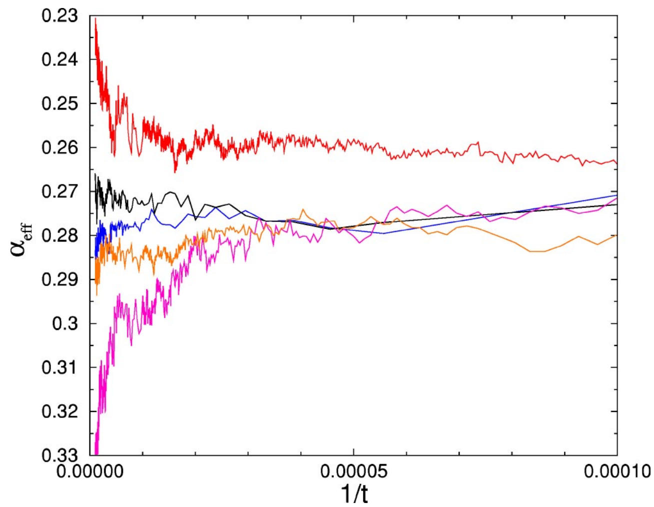

FIG. 10. Local slopes of the density decay in the one-dimensional $4A \to 5A$, $4A \to \emptyset$ model for $D=0.3$. Different curves correspond to $p=0.904$, 0.9037, 0.9033, 0.903 0.9027, and 0.902 (from bottom to top).

C. $4A \to 5A$, $4A \to \emptyset$ and $4A \to 5A$, $2A \to \emptyset$ quadruplet models

Two-dimensional simulations (Sec. III), showed that for $n=m=4$ symmetric quadruplet models, $d_c=2$. Simulations in the corresponding suppressed bosonic SCA [18] with $n=4$ and $1 \leqslant m \leqslant 4$ located the phase transition at zero production rate. Here, I show that in the one-dimensional $4A \to 5A$, $4A \to \emptyset$ and $4A \to 5A$, $2A \to \emptyset$ site restricted models, continuous phase transitions with $p<1$ and with nontrivial exponents can be found. The density decay was followed up to $t=10^6$ MCS and the critical point was located by the local-slopes method (see Fig. 10) at $p=0.9028(1)$ for $D=0.3$. The corresponding exponent can be estimated as $\alpha=0.27(1)$. For $D=0.05$, one gets $p_c=0.9605(3)$ with $\alpha=0.28(1)$, so one cannot see diffusion dependence here. Analyzing off-critical data with the local slopes method (18), one gets $\beta=0.48(2)$ (see Fig. 7).

In accordance with these results, simulations for the $4A \to 5A$, $2A \to \emptyset$ model at $D=0.2$ and $D=0.8$ diffusion rates, resulted in $p_c(0.2)=0.531\,85(5)$ and $p_c(0.8)=0.5742(1)$ with $\alpha=0.27(1)$ and $\beta=0.48(2)$ exponents (see Fig. 7). As we can see, critical exponent data for quadruplet models are robust and no diffusion dependence has been found. Furthermore, critical space-time plots are very similar to that of the PCPD model.

V. CONCLUSIONS

In this paper, I investigated the phase transitions of general $nA \to (n+k)A$, $mA \to (m-l)A$ reaction type of models with explicit single particle diffusion on occupation number restricted lattices in one and two dimensions. I showed that the mean-field solution for $n=m$ symmetric cases results in universality classes characterized by the exponents $\alpha=1/n$ and $\beta=1$. I determined the upper critical dimensions for the triplet and quadruplet cases by simulations. For $n=3$, high precision simulations show the mean-field type of criticality with logarithmic corrections meaning $d_c=1$. This result is in contradiction with the simulations of Refs. [17] and [18] and with the analytical form for $d_c(n)$ derived from a phenomenological Langevin equation. In case of my site restricted realization of the one-dimensional $3A \to 6A$, $3A \to \emptyset$ model, the phase transition point is shifted to zero production rate and is continuous, BkARW mean-field type. This is in contradiction with the findings of Ref. [18] for another stochastic cellular automaton realization of this model. For $n=4$ the upper critical dimension was located at $d_c=2$, opening up the possibility for nontrivial critical behavior in $d=1$. Indeed, two versions of such quadruplet models were shown to exhibit robust, novel type of critical transition in one dimension.

For $n>m$, the mean-field approximation gives first-order transition that was observed by simulations for two ($n=3,4$) models in $d=2$. On the other hand, numerical evidence was given that the parity conserving model $3A \to 5A$, $2A \to \emptyset$ in one dimension exhibits a non-PC type of critical behavior with logarithmic corrections by varying the diffusion rate. This transition does not fit in the universality class scheme suggested by Ref. [18].

Finally, I showed that for $n<m$ models, the mean-field approximations result in classes that feature $\alpha^{MF}=1/(m-1)$ and $\beta^{MF}=1/(m-n)$. Such kinds of models should be the subject of further studies.

The presented mean-field and simulation results show that the universal behavior of such low-dimensional reaction-diffusion models is rich and the table of universality classes given by Ref. [18] is not valid for one-dimensional, fully site restricted systems. Perhaps, the strict site restriction plays an important role that causes the differences. Field theoretical (possibly fermionic) treatment that starts from the master equation should be set up to determine at least the analytical form of $d_c(n)$ for $n=m>2$ models.

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