Universality classes in nonequilibrium lattice systems

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This article reviews our present knowledge of universality classes in nonequilibrium systems defined on regular lattices. The first section presents the most important critical exponents and relations, as well as the field-theoretical formalism used in the text. The second section briefly addresses the question of scaling behavior at first-order phase transitions. In Sec. III the author looks at dynamical extensions of basic static classes, showing the effects of mixing dynamics and of percolation. The main body of the review begins in Sec. IV, where genuine, dynamical universality classes specific to nonequilibrium systems are introduced. Section V considers such nonequilibrium classes in coupled, multicomponent systems. Most of the known nonequilibrium transition classes are explored in low dimensions between active and absorbing states of reaction-diffusion-type systems. However, by mapping they can be related to the universal behavior of interface growth models, which are treated in Sec. VI. The review ends with a summary of the classes of absorbing-state and mean-field systems and discusses some possible directions for future research.

CONTENTS

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I. INTRODUCTION

Universal scaling behavior is an attractive feature in statistical physics because a wide range of models can be classified purely in terms of their collective behavior. Scaling phenomena have been observed in many branches of physics, chemistry, and biology, as well as in economics and the social sciences, most frequently by critical phase transitions. For example, nonequilibrium phase transitions may appear in models of population (Albano, 1994), epidemics (Mollison, 1977; Ligget, 1985), catalysis (Ziff *et al.*, 1986), cooperative transport (Havlin and ben Avraham, 1987; Chowdhury *et al.*, 2000), enzyme biology (Berry, 2003), and markets (Bouchaud and Georges, 1990).

Dynamical extensions of static universality classes—established in equilibrium—are the simplest nonequilibrium model systems, but beyond that critical phenomena, with new classes, have also been explored (Grassberger, 1996; Marro and Dickman, 1999; Hinrichsen, 2000a). While the theory of phase transitions is quite well understood in thermodynamic equilibrium, its use in nonequilibrium systems is rather new. In these systems most of the fundamental concepts of equilibrium models—phase transitions, scaling, and universality—still apply. The basic ingredients affecting universality classes are again the collective behavior of systems, the symmetries, the conservation laws, and the spatial dimensions, as described by renormalization-group theory. But in addition several new factors have been identified. Low-dimensional systems are of particular interest because in them fluctuation effects are relevant, hence the mean-field type of description is not valid. Over the past two decades this field of research has grown very quickly and now we are faced with a wide variety of models and universality classes, labeled by strange notations and abbreviations. This article aims to help newcomers as well as experienced researchers navigate the literature by systematically reviewing most of the explored universality classes. I define models by their field theory (when it is available), show their symmetries or other important features, and list the critical exponents and scaling relations.

Nonequilibrium systems can be classified into two categories:

(a) Systems that have a Hermitian Hamiltonian and whose stationary states are given by the proper Gibbs-Boltzmann distribution. These, however, are prepared under initial conditions far from the stationary state. Sometimes, in the thermodynamic limit, such systems may never reach true equilibrium. Examples include phase-ordering systems, spin glasses, and glasses. I begin the review of classes by showing the scaling behavior of the simplest prototypes of such systems in Sec. III, defining them by adding simple dynamics to static models.

(b) Systems without a Hermitian Hamiltonian defined by transition rates, which do not satisfy the detailed balance condition (the local time-reversal symmetry is broken). They may or may not have a steady state, and even if they have one, it is not a Gibbs state. Such models can be created by combining different dynamics or by generating currents in them externally. The critical phenomena of these systems are referred to here as "*out-of-equilibrium classes*" and discussed in Sec. III.

There are also systems that are not related to equilibrium models. In the simplest case, these are lattice Markov processes of interacting particle systems (Ligget, 1985), referred to here as "*genuine nonequilibrium systems*." These are discussed in the rest of the work. The discussion of the latter type of system is divided into three parts: In Sec. IV phase-transition classes, the simplest of such models, are presented. These are usually *reaction-diffusion models* exhibiting a phase transition to an absorbing state. In Sec. V, I list the known classes that occur as combinations of basic genuine class processes. These models are coupled multicomponent reaction-diffusion systems. While the former two sections are related to critical phenomena near extinction, in Sec. VI I discuss universality classes in systems where site variables are nonvanishing, in surface growth models. The bosonic field-theoretical description is applicable to them as well. I point out mapping between growth and reaction-diffusion systems whenever possible. In Sec. II, I briefly touch on the discontinuous nonequilibrium phase transitions, especially because dynamical scaling may occur at such points.

I define a *critical universality class* by the complete set of exponents at the phase transition. Therefore different dynamics split up the basic static classes of homogeneous systems. I emphasize the role of symmetries and boundary conditions that affect these classes. I also point out very recent evidence according to which in low-dimensional systems symmetries are not necessarily the most relevant factors of universality classes. Although the systems covered here might seem artificial to experimentalists or to applications-oriented people, they constitute the building blocks of our understanding of nonequilibrium critical phenomena. Understanding even these simple models can often pose tremendously difficult problems.

I shall not discuss the critical behavior of quantum systems (Rácz, 2002) or self-organized critical phenomena (Bak *et al.*, 1987), nor shall I consider experimental realizations or applied methods, due to the lack of space, although in Sec. I.E I give a brief introduction to the field-theoretical approach. This section presents the formalism for defining nonequilibrium models, which is necessary to express the symmetry relations affecting critical behavior. Researchers from other branches of science are provided a kind of catalog of classes in which they can identify their models and find corresponding theories. A list of the most common abbreviations is given at the end of the text.

There are many interesting features of universality classes that I do not discuss in this review—scaling functions, fluctuation distributions, extremal statistics, finite-size effects, statistics of fluctuations in surface growth models, etc. Still I believe the material included provides a useful framework for orientation in this huge field. There is no general theory of nonequilibrium phase transitions; hence a broad overview of known classes can help theorists to identify the relevant factors in determining universality classes.

There are two recent, similar reviews available. One of them is by Marro and Dickman (1999), which gives a pedagogical introduction to driven lattice-gas systems and to fundamental-particle systems with absorbing states. The other, by Hinrichsen (2000a) focuses more on basic absorbing-state phase transitions, methods, and experimental realizations. However the field is evolving rapidly, and since the publication of these two noteworthy introductory works a series of new developments have come up. The present work aims to give a comprehensive overview of known nonequilibrium dynamical classes, incorporating surface growth classes, classes of spin models, percolation and multicomponent system classes, and damage-spreading transitions. The relations and mappings of the corresponding models are pointed out. The effects of boundary conditions, long-range interactions, and disorder are shown systematically for each class when these are known. Since the debate on the conditions of the parity-conserving class has not yet been settled I discuss it, using some surface-catalytic model examples. Naturally a review of this scope cannot cover the literature completely, and I apologize for the omitted references.

A. Critical exponents of equilibrium systems

In this section I offer brief definitions of well-known critical exponents of homogeneous equilibrium systems and show some scaling relations (Fisher, 1967; Kadanoff *et al.*, 1967; Stanley, 1971; Ma, 1976; Amit, 1984). The basic exponents are defined via the following scaling laws:

$$c_H \propto \alpha_H^{-1}[(|T-T_c|/T_c)^{-\alpha_H}-1], \tag{1}$$

$$m \propto (T_c - T)^{\beta}, \tag{2}$$

$$\chi \propto |T-T_c|^{-\gamma}, \tag{3}$$

$$m \propto H^{1/\delta_H}, \tag{4}$$

$$G_c^{(2)}(r) \propto r^{2-d-\eta_\alpha}, \tag{5}$$

$$\xi \propto |T-T_c|^{-\nu_\perp}. \tag{6}$$

Here c_H denotes the specific heat, m the order parameter, χ the susceptibility, and ξ the correlation length. The presence of another degree of freedom besides the temperature T, like a (small) external field (labeled by H), leads to other interesting power laws when $H \to 0$. The d present in the expression for the two-point correlation function $G_c^{(2)}(r)$ is the space dimension of the system.

Some laws are valid both to the right and to the left of the critical point; the values of the relative proportionality constants, or *amplitudes*, are in general different for the two branches of the functions, whereas the exponent is the same. However, there are universal amplitude relations among them. We can see that there are altogether six basic exponents. Nevertheless, they are not independent of each other, but related by some simple scaling relations:

$$\alpha_H + 2\beta + \gamma = 2, \quad \alpha_H + \beta(\delta_H + 1) = 2, \tag{7}$$

$$(2-\eta_a)\nu_\perp = \gamma, \quad \nu_\perp d = 2 - \alpha_H.$$

The last relation is a so-called hyperscaling law, which depends on the spatial dimension d and, according to the Gaussian theory, is not valid above the upper critical dimension d_c. Therefore below d_c there are only two independent exponents in equilibrium. One of the most interesting aspects of second-order phase transitions is their so-called *universality*, i.e., the fact that systems very different from each other can share the same set of critical indices (exponents and some amplitude ratios). One can therefore hope to assign all systems to *classes*, each of which is identified by a set of critical indices.

B. Static percolation cluster exponents

Universal behavior may occur as percolation (Stauffer and Aharony, 1994; Grimmett, 1999), which can be considered a purely geometrical phenomenon describing the occurrence of infinitely large connected clusters on lattices. On the other hand, such clusters emerge at the critical phase transitions of lattice models. The definition of a connected cluster is not unambiguous. It may mean a set of sites or bonds with variables in the same state, or sites connected by bonds with probability $b=1-\exp(-2J/kT)$.

When the system control parameter p, often the temperature in equilibrium systems, is tuned to the critical value p_p, the coherence length between sites may diverge as

$$\xi(p) \propto |p-p_p|^{-\nu_\perp}. \tag{8}$$

Hence percolation at p_p, as in standard critical phenomena, exhibits renormalizibility and universality of critical exponents. At p_p the distribution of the cluster size s follows the scaling law

$$n_s \propto s^{-\tau} f(|p-p_p| s^{\sigma}), \tag{9}$$

while moments of this distribution exhibit singular behavior with the exponents

$$\sum_s s n_s(p) \propto |p-p_p|^{\beta_p}, \tag{10}$$

$$\sum_s s^2 n_s(p) \propto |p-p_p|^{-\gamma_p}. \tag{11}$$

Further critical exponents and the scaling relations among them are shown by Stauffer and Aharony (1994). In the case of completely random placement of sites, bonds, etc. (with probability p) on lattices we find *random isotropic (ordinary) percolation* (see Sec. IV.B.1). Percolating clusters may arise at critical, thermal transitions or by nonequilibrium processes. If the critical point (p_c) of the percolation order parameter does not coincide with p_p then at the percolation transition the coherence length of the order parameter is finite and does not influence the percolation properties. We observe random percolation in that case. In contrast, if $p_p=p_c$ percolation is influenced by the order parameter behavior and we find *correlated percolation* universality (Fortuin and Kasteleyn, 1972; Coniglio and Klein, 1980; Stauffer and Aharony, 1994) whose exponents may coincide with those of the order parameter.

According to the Fortuin-Kasteleyn construction of clusters (Fortuin and Kasteleyn, 1972), two nearest-neighbor spins of the same state belong to the same cluster with probability $b=1-\exp(-2J/kT)$. It was shown that when one uses this prescription for Z_n and $O(n)$ and symmetric models (Coniglio and Klein, 1980; Bialas *et al.*, 2000; Blanchard *et al.*, 2000; Fortunato and Satz, 2001) the *thermal phase-transition* point coincides with the percolation limits of such clusters. On the other hand, in the case of "pure-site clusters" ($b=1$), different, universal cluster exponents are reported in two-dimensional models (Fortunato and Satz, 2001; Fortunato, 2002; see Secs. III.A.1, III.B.1, and III.D).

C. Dynamical critical exponents

Nonequilibrium systems were first introduced to study relaxation in equilibrium states (Halperin and Hohenberg, 1977) and phase-ordering kinetics (Binder and Stauffer, 1974; Marro *et al.*, 1979). Power-law time dependences were investigated away from the critical point, as well, for example, in the domain growth in a quench to $T=0$. Later the combination of different heat baths, different dynamics, and external currents became popular tools for the investigation of fully nonequilibrium models. To describe the dynamical behavior of a critical system additional exponents were introduced. For example, the divergences of the relaxation time τ and correlation length ξ are related by the dynamical exponent Z,

$$\tau \propto \xi^{Z}. \tag{12}$$

Systems out of equilibrium may show anisotropic scaling of two (and n) point functions,

$$G(b\mathbf{r}, b^{\zeta} t) = b^{-2x} G(\mathbf{r}, t), \tag{13}$$

where $\mathbf{r}$ and t denote spatial and temporal coordinates, x is the scaling dimension, and ζ is the anisotropy exponent. As a consequence the temporal ($\nu_\parallel$) and spatial ($\nu_\perp$) correlation length exponents may be different, described by $\zeta=Z$:

$$Z=\zeta=\frac{\nu_\parallel}{\nu_\perp}. \tag{14}$$

For some years it was believed that dynamical critical phenomena are characterized by a set of three critical exponents, comprising two independent static exponents (other static exponents being related to these by scaling laws) and the dynamical exponent Z. Recently it was discovered that there is another dynamical exponent, the nonequilibrium or *short-time exponent* λ, needed to describe two-time correlations in a spin system $\{s_i\}$ of size L relaxing to the critical state from a disordered initial condition (Huse, 1989; Janssen *et al.*, 1989):

$$A(t,0)=\frac{1}{L^d}\left\langle \sum_i s_i(0) s_i(t) \right\rangle \propto t^{-\lambda/Z}. \tag{15}$$

More recently the *persistence exponents* θ_l and θ_g were introduced by Derrida *et al.* (1994; see also Majumdar *et al.*, 1996). These are associated with the probability $p(t)$ that the local or global order parameter has not changed sign in time t following a quench to the critical point. In many systems of physical interest these exponents decay algebraically as

$$p(t) \propto t^{-\theta} \tag{16}$$

(see, however, Sec. V.A). It turns out that in systems where the scaling relation

$$\theta_g Z = \lambda - d + 1 - \eta_a/2 \tag{17}$$

is satisfied the dynamics of the global order parameter is a Markov process. In contrast, in systems with non-Markovian global order parameter, θ_g is in general a new, nontrivial critical exponent (Majumdar *et al.*, 1996). For example, it was shown that while in the $d=1$ Glauber Ising model the magnetization is Markovian and the scaling relation (17) is fulfilled, at the critical point of the $d=1$ nonequilibrium kinetic Ising model condition (17) is not satisfied and the persistence behavior there is characterized by a different, nontrivial θ_g exponent (Menyhárd and Ódor, 1997; see discussion in Sec. IV.D.2). As we can see, the universality classes of static models are distinguished by their dynamical exponents.

D. Critical exponents of spreading processes

In the previous section I defined quantities describing dynamical properties of the bulk material. Alternatively one may also consider cluster properties arising from an ordered (correlated) state with a small cluster of activity. Here I define a basic set of critical exponents that occur in spreading processes and show the scaling relations among them. In such processes there may be a phase transition to an absorbing state where the density of the spreading entity (particle, agent, epidemic, etc.) goes to zero. The order parameter is usually the density of active sites $\{s_i\}$,

$$\rho(t) = \frac{1}{L^d}\left\langle \sum_i s_i(t) \right\rangle, \tag{18}$$

which in the supercritical phase vanishes as

$$\rho^\infty \propto |p - p_c|^\beta, \tag{19}$$

as the control parameter p is varied. Another quantity is the ultimate survival probability P_∞ of an infinite cluster of active sites that scales in the active phase as

$$P_\infty \propto |p - p_c|^{\beta'} \tag{20}$$

with some critical exponent β' (Grassberger and de la Torre, 1979). In field-theoretical descriptions of such processes, β is associated with the particle annihilation and β' with the particle creation operator, and in the case of time-reversal symmetry [see Eq. (88)] they are equal. The critical long-time behavior of these quantities is described by

$$\rho(t) \propto t^{-\alpha} f(\Delta t^{1/\nu_\parallel}), \quad P(t) \propto t^{-\delta} g(\Delta t^{1/\nu_\parallel}), \tag{21}$$

where α and δ are the critical exponents for decay and survival, $\Delta = |p - p_c|$, f and g are *universal scaling functions* (Grassberger and de la Torre, 1979; Muñoz *et al.*, 1997; Janssen, 2003). The obvious scaling relations among them are

$$\alpha = \beta/\nu_\parallel, \quad \delta = \beta'/\nu_\parallel. \tag{22}$$

For finite systems (of size $N = L^d$) these quantities scale as

$$\rho(t) \propto t^{-\beta/\nu_\parallel} f'(\Delta t^{1/\nu_\parallel}, t^{d/Z}/N), \tag{23}$$

$$P(t) \propto t^{-\beta'/\nu_\parallel} g'(\Delta t^{1/\nu_\parallel}, t^{d/Z}/N). \tag{24}$$

For relatively short times or for initial conditions with a single active seed, the number of active sites $N(t)$ and its mean square spreading distance R from the origin

$$R^2(t) = \frac{1}{N(t)}\left\langle \sum_i x_i^2(t) \right\rangle \tag{25}$$

follow the "initial slip" scaling laws (Grassberger and de la Torre, 1979)

$$N(t) \propto t^\eta, \tag{26}$$

$$R^2(t) \propto t^z, \tag{27}$$

and usually the relation $z = 2/Z$ holds.

Phase transitions between chaotic and nonchaotic states may be described as *damage spreading* (DS). While DS was first introduced in biology (Kauffman, 1969) it has become an interesting topic in physics as well (Creutz, 1986; Stanley *et al.*, 1986; Derrida and Weisbuch, 1987). The main question is whether damage introduced in a dynamical system survives or disappears. To investigate this, the usual technique is to make one or more replicas of the original system and let them evolve with the same dynamics and external noise. This method has been found to be very useful for accurate measurements of the dynamical exponents of equilibrium systems (Grassberger, 1995a). It has turned out, however, that DS properties do depend on the applied dynamics. An example is the case of the two-dimensional Ising model with heat-bath algorithm or with Glauber dynamics (Mariz *et al.*, 1990; Jan and de Arcangelis, 1994; Grassberger, 1995b).

To avoid dependency on the dynamics, a definition of a "physical" family of DS dynamics was suggested by Hinrichsen *et al.* (1997) in which the active phase may be divided into three subphases: a subphase in which damage occurs for every member of the family, another subphase in which the damage heals for every member of the family, and a third possible subphase in which damage is possible for some members and disappears for other members. The family of possible DS dynamics is defined to be consistent with the physics of single replicas (symmetries, interaction ranges, etc.).

Usually the order parameter of the damage is the Hamming distance between replicas,

$$D(t) = \left\langle \frac{1}{L}\sum_{i=1}^{L} |s(i) - s'(i)| \right\rangle, \tag{28}$$

where $s(i)$ and $s'(i)$ denote variables of the replicas. At continuous DS transitions D exhibits power-law singularities as physical quantities at the critical point. For example, one can follow the fate of a single difference between two (or more) replicas and measure the spreading exponents:

$$D(t) \propto t^{\eta_d}. \tag{29}$$

Similarly damage variables have the survival probability

$$P_D(t) \propto t^{-\delta_d} \tag{30}$$

and similarly to Eq. (25) the average mean-square spreading distance of a damage variable from the center scales as

$$R_D^2(t) \propto t^{z_d}. \tag{31}$$

Grassberger conjectured that all DS transitions should belong to the directed percolation class (see Sec. IV.A) unless they coincide with other transition points and that the probability for a locally damaged state to become healed is nonzero (Grassberger, 1995c). This hypothesis has been confirmed by simulations of many different systems.

E. Field-theoretical approach to reaction-diffusion systems

In this review I define nonequilibrium systems formally by their field-theoretical action wherever possible. Therefore in this subsection I give a brief introduction to the (bosonic) field-theoretical formalism. This will be through the simplest example of reaction-diffusion systems, via the $A+A\to\emptyset$ annihilating random walk (ARW; see Sec. IV.C.1). A similar stochastic differential equation can also be set up for growth processes in most cases. For a more complete introduction see Cardy (1996, 1997) and Täuber (2003).

A proper field-theoretical treatment should start from the master equation for the microscopic time evolution of probabilities $p(\alpha;t)$ of states α,

$$\frac{dp(\alpha;t)}{dt} = \sum_{\beta} R_{\beta\to\alpha}p(\beta;t) - \sum_{\beta} R_{\alpha\to\beta}p(\alpha;t), \tag{32}$$

where $R_{\alpha\to\beta}$ denotes the transition matrix from state α to state β. In field theory this can be expressed in a Fock-space formalism with annihilation (a_i) and creation (c_i) operators satisfying the commutation relation

$$[a_i,c_j] = \delta_{ij}. \tag{33}$$

The states are built up from the vacuum $|0\rangle$ as the linear superposition

$$\Psi(t) = \sum_{\alpha} p(n_1,n_2,\ldots;t)c_1^{n_1}c_2^{n_2}\cdots|0\rangle, \tag{34}$$

with occupation number coefficients $p(n_1,n_2,\ldots;t)$. The evolution of states can be described by a Schrödinger-like equation,

$$\frac{d\Psi(t)}{dt} = -H\Psi(t), \tag{35}$$

with a generally non-Hermitian Hamiltonian, which in case of the ARW process looks like

$$H = D\sum_{ij}(c_i - c_j)(a_i - a_j) - \lambda\sum_{j}(a_j^2 - c_j^2a_j^2). \tag{36}$$

Here D denotes the diffusion strength and λ the annihilation rate. By going to the continuum limit this turns into

$$H = \int d^dx[D(\nabla\psi)(\nabla\phi) - \lambda(\phi^2 - \psi^2\phi^2)], \tag{37}$$

and in the path-integral formalism over fields $\phi(x,t)$, $\psi(x,t)$ with weight $e^{-S(\phi,\psi)}$ one can define an action, that in case of an ARW is

$$S = \int dtd^dx[\psi\partial_t\phi + D\nabla\psi\nabla\phi - \lambda(\phi^2 - \psi^2\phi^2)]. \tag{38}$$

The action is analyzed by renormalization-group (RG) methods at criticality (Ma, 1976; Amit, 1984), usually by perturbative epsilon expansion below the upper critical dimension d_c—that is, the lower limit of the validity of the mean-field behavior of the system. The symmetries of the model can be expressed in terms of the $\phi(x,t)$ field and $\psi(x,t)$ response field variables, and the corresponding hyperscaling relations can be derived (Muñoz *et al.*, 1997; Janssen, 2003).

By a Gaussian transformation one may set up an alternative formalism—integrating out the response field—the Langevin equation, which in the case of an ARW is

$$\partial_t\phi = D\nabla^2\phi - 2\lambda\phi^2 + \eta(x,t), \tag{39}$$

with a Gaussian noise, exhibiting the correlations

$$\langle\eta(x,t)\eta(x',t')\rangle = -\lambda\phi^2\delta(x-x')\delta(t-t'). \tag{40}$$

Here δ denotes the Dirac delta function and λ is the noise amplitude. From the Langevin equation—if it applies—one can deduce a naive upper critical dimension (d_c) by power counting. However, this estimate may be modified by fluctuations, which can be analyzed by application of the RG method.

II. SCALING AT FIRST-ORDER PHASE TRANSITIONS

In nonequilibrium systems dynamical scaling of variables may occur even when the order parameter jumps at the transition. We call such a transition first order, although the free energy is not defined. First-order phase transitions have rarely been seen in low dimensions. This is due to the fact that in lower dimension

TABLE I. Critical one-dimensional Ising spin exponents at the Glauber and nonequilibrium kinetic Ising model (NEKIM) transition points. From Menyhárd and Ódor, 1998.

	β_s	$\nu_\perp$	β_s'	Δ	η_s	δ_s	z_s
Glauber	0	1/2	0.99(2)	1/2	0.0006(4)	0.500(5)	1
NEKIM	0.00(2)	0.444	0.45(1)	0.49(1)	0.288(4)	0.287(3)	1.14

fluctuations are relevant and may destabilize the ordered phase. Therefore fluctuation-induced second-order phase transitions are likely to appear. Hinrichsen advanced the hypothesis (Hinrichsen, 2000b) that first-order transitions do not exist in (1+1)-dimensional systems without extra symmetries, conservation laws, special boundary conditions, or long-range interactions (which can be generated by macroscopic currents or anomalous diffusion in nonequilibrium systems, for instance). Examples are the Glauber and the nonequilibrium kinetic Ising spin systems (see Secs. III.A and IV.D.2) possessing Z_2 symmetry in one dimension (Glauber, 1963; Menyhárd and Ódor, 1998), where the introduction of a "temperaturelike" flip inside of a domain or an external field (h) causes a discontinuous jump in the magnetization order parameter (m). Interestingly enough the correlation length diverges at the transition point: $\xi \propto p_T^{-\nu_\perp}$ and static magnetization

$$m \propto \xi^{-\beta_s/\nu_\perp} g(h\xi^{\Delta/\nu_\perp}) \tag{41}$$

as well as cluster critical exponents can be defined:

$$P_s(t,h) \propto t^{-\delta_s}, \tag{42}$$

$$R_s^2(t,h) \propto t^{z_s}, \tag{43}$$

$$|m(t,h) - m(0)| \propto t^{\eta_s}, \tag{44}$$

$$\lim_{t\to\infty} P_s(t,h) \propto h^{\beta_s'}. \tag{45}$$

Here s refers to the spin variables. Table I summarizes the results obtained for these transitions. Other examples of first-order transitions are known in driven diffusive systems (Janssen and Schmittman, 1986), in the one-dimensional asymmetric exclusion process (Derrida, 1998), in bosonic annihilation-fission models (Sec. V.F), in asymmetric triplet and quadruplet models (Ódor, 2003a; Sec. IV.F) and diffusive conserved-field models for $D_A > D_B$ and $d > 1$ (Oerding *et al.*, 2000; Sec. V.H). It is quite difficult to decide by simulations whether a transition is really discontinuous. The order parameter of weak first-order transitions—where the jump is small—may look very similar to continuous transitions. Measuring the hysteresis of the order parameter that is considered to be the indication of a first-order transition is a demanding task. For examples of debates over the order of the transition, see Tomé and de Oliveira (1989); Dickman and Tomé (1991); Lipowski (1999); Lipowski and Lopata (1999); Hinrichsen (2000b, 2001a); Szolnoki (2000). In some cases the mean-field solution results in a first-order transition (Ódor *et al.*, 1993; Menyhárd and Ódor, 1995). In two dimensions there are certain stochastic cellular automata for which systematic cluster mean-field techniques combined with simulations have made it possible to prove first-order transitions firmly (Ódor and Szolnoki, 1996; see Table II).

III. OUT-OF-EQUILIBRIUM CLASSES

In this section I begin by introducing basic nonequilibrium classes, starting with the simplest dynamical extensions of equilibrium models. These dynamical systems exhibit Hermitian Hamiltonians and, starting from a nonequilibrium state, they evolve into a Gibbs state. Such nonequilibrium models include, for example, phase-ordering systems, spin glasses, glasses, etc. In these cases one is usually interested in the nonequilibrium dynamics at the equilibrium critical point.

It is important to note that scaling behavior can be observed far away from criticality, as well. In quenches to zero temperature of *model-A systems* (which do not conserve the order parameter), the characteristic length in the late-time regime grows with a universal power law $\xi \propto t^{1/2}$, while in the case of *model-B systems* (which conserve the order parameter), $\xi \propto t^{1/3}$. In *model-C systems* a conserved secondary density is coupled to the nonconserved order parameter. Such models may exhibit model-A behavior or $\xi \propto t^{1/(2+\alpha_H/\nu_\perp)}$ depending on the model parameters. The effects of such conservation laws in critical systems without Hermitian Hamiltonians have also been investigated and will be discussed in later sections.

Since percolation is a central topic in reaction-diffusion systems, discussed in Secs. IV and V, for the sake of completeness I show recent percolation results

TABLE II. Convergence of the critical point estimates in various ($y=1,2,3$) two-dimensional stochastic cellular automata calculated by n-cluster mean-field approximation (see Sec. IV.A.2). First-order transitions are denoted by boldface numbers. The gap sizes $[\rho(p_c)]$ of the order parameter shown for $y=2,3$ increase with n, approximating the simulation value. From Ódor and Szolnoki, 1996.

	$y=1$	$y=2$		$y=3$	
n	p_c	p_c	$\rho(p_c)$	pc	$\rho(p_c)$
1	0.111	**0.354**	**0.216**	**0.534**	**0.372**
2	0.113	**0.326**	**0.240**	**0.455**	**0.400**
4	0.131	**0.388**	**0.244**	**0.647**	**0.410**
Simulation	0.163	**0.404**	**0.245**	**0.661**	**0.418**

obtained for systems discussed in this section, too. Another novel feature of dynamic phase transitions is the emergence of a chaotic state. Therefore I shall discuss damage-spreading transitions and behavior in these systems.

Then I consider nonequilibrium models that have no Hermitian Hamiltonian and no equilibrium Gibbs state. I show cases when this is achieved by combining different competing dynamics (for example, by connecting two reservoirs with different temperatures to the system) or by generating current from outside. Field-theoretical investigations have revealed that model-A systems are robust against the introduction of various competing dynamics, which are local and do not conserve the order parameter (Grinstein *et al.*, 1985). Furthermore it has been shown that this robustness of the critical behavior persists if the competing dynamics breaks the discrete symmetry of the system (Bassler and Schmittman, 1994) or if it comes from reversible mode coupling to a noncritical conserved field (Täuber and Rácz, 1997). On the other hand, if competing dynamics are coupled to model-B systems by an external drive (Schmittman and Zia, 1996) or by a local, anisotropic order-parameter-conserving process (Schmittman and Zia, 1991; Schmittman, 1993; Bassler and Rácz, 1994, 1995) long-range interactions are generated in the steady state with angular dependence. The universality class will be the same as that of the kinetic version of the equilibrium Ising model with dipolar long-range interactions.

As the number of neighboring interaction sites decreases, which can happen when the spatial dimensionality of a system with short-ranged interactions is lowered, the importance of fluctuations increases. In equilibrium models finite-range interactions cannot maintain long-range order in $d<2$. This observation is known as the Landau-Peierls argument (Landau and Lifshitz, 1981). According to the Mermin-Wagner theorem (Mermin and Wagner, 1996), for systems with continuous symmetry, long-range order does not exist even in $d=2$. Hence in equilibrium models phase-transition universality classes exist for $d\geqslant 2$ only. One of the main open questions to be answered is whether there exists a class of nonequilibrium systems with restricted dynamical rules for which the Landau-Peierls or Mermin-Wagner theorem can be applied.

A. Ising classes

The equilibrium Ising model was introduced by Ising (1925) as the simplest model for a uniaxial magnet, but it is used in different settings, for example, binary fluids or alloys, as well. It is defined in terms of spin variables $s_i=\pm 1$ attached to sites i of some lattice with the Hamiltonian

$$H=-J\sum_{i,i'} s_i s_{i'} - B\sum_i s_i, \tag{46}$$

where J is the coupling constant and B is the external field. In one and two dimensions it is solved exactly (Onsager, 1944), hence it plays a fundamental role as a test ground for understanding phase transitions. The Hamiltonian of this model exhibits a global, so-called Z_2 (up-down) symmetry of the state variables. While in one dimension a first-order phase transition occurs at $T=0$ only (see Sec. II), in two dimensions there is a continuous phase transition where the system exhibits conformal symmetry (Henkel, 1999). The critical dimension is $d_c=4$. Table III summarizes some of the known critical exponents of the Ising model. The quantum version of the Ising model, which in the simplest cases might take the form (in one dimension)

TABLE III. Static critical exponents of the Ising model.

Exponent	$d=2$	$d=3$	$d=4$ (MF)
α_H	0(ln)	0.1097(6)	0
β	1/8	0.3265(7)	1/2
γ	7/4	1.3272(3)	1
$\nu_\perp$	1	0.6301(2)	1/2

$$H=-J\sum_i (t\sigma_i^x+\sigma_i^z\sigma_{i+1}^z+h\sigma_i^z), \tag{47}$$

where $\sigma^{x,z}$ are Pauli matrices and t and h are couplings—for $T>0$ has been shown to exhibit the same critical behavior as the classical version (in the same dimension). For $T=0$, however, quantum effects become important and the quantum Ising chain can be associated with the two-dimensional classical Ising model, with the transverse field t playing the role of the temperature. In general a mapping can be constructed between classical $(d+1)$-dimensional statistical systems and d-dimensional quantum systems without changing the universal properties. This mapping has been widely utilized (Suzuki, 1971; Fradkin and Susskind, 1978). The effects of disorder and boundary conditions are not discussed here (for recent reviews see Iglói *et al.*, 1993; Alonso and Munoz, 2001).

1. Correlated percolation clusters at T_c

If we generate clusters in such a way that we join nearest-neighbor spins of the same sign we can observe percolation at T_c in two dimensions. While the order-parameter percolation exponents β_p and γ_p of this percolation (defined in Sec. IV.B.1) are different from the exponents of the magnetization (β,γ), the correlation length exponent is the same: $\nu=\nu_p$. For two-dimensional models with Z_2 symmetry, the universal percolation exponents are (Fortunato and Satz, 2001; Fortunato, 2002)

$$\beta_p=0.049(4), \quad \gamma_p=1.908(16). \tag{48}$$

These exponents are clearly different from those of the ordinary percolation classes (Table XV) or from Ising-class magnetic exponents (Table III).

TABLE IV. Critical dynamical exponents in the Ising model. Column headings (A) and (B) refer to model-A and model-B dynamics. Data are from Grassberger (1995a), Jaster *et al.* (1999), Stauffer (1996), Zheng (1998, 2000, 2001) and Zwerger (1981).

	$d=1$		$d=2$		$d=3$	$d=4$	
	A	B	A	B	A	A	B
Z	2	5	2.165(10)	2.325(10)	2.032(4)	2	4
λ	1		0.737(1)	0.667(8)	1.362(19)	4	
θ_g	1/4		0.225(10)		0.41(2)	1/2	

On the other hand, by Fortuin-Kasteleyn cluster construction (Fortuin and Kasteleyn, 1972), the percolation exponents of the Ising model at T_c coincide with those of the magnetization of the model.

2. Dynamical Ising classes

Kinetic Ising models such as the spin-flip Glauber Ising model (Glauber, 1963) and the spin-exchange Kawasaki Ising model (Kawasaki, 1966) were originally intended to study relaxational processes near equilibrium states. In order to assure arrival at an equilibrium state, the detailed balance condition for transition rates ($w_{i\to j}$) and probability distributions $[P(s)]$ must satisfy

$$w_{i\to j}P[s(i)] = w_{j\to i}P[s(j)]. \tag{49}$$

Knowing that $P_{eq}(s)\propto \exp[-H(s)/(k_B T)]$, this entails the condition

$$\frac{w_{i\to j}}{w_{j\to i}} = \exp[-\Delta H(s)/(k_B T)], \tag{50}$$

which can be satisfied in many different ways. Assuming spin-flips [which do not conserve the magnetization (model A)] Glauber formulated the most general form in a magnetic field (h),

$$w_i^h = w_i(1-\tanh hs_i) \approx w_i(1-hs_i), \tag{51}$$

$$w_i = \frac{\Gamma}{2}(1+\tilde{\delta}s_{i-1}s_{i+1})\left(1-\frac{\gamma}{2}s_i(s_{i-1}+s_{i+1})\right), \tag{52}$$

where $\gamma=\tanh 2J/kT$, Γ and $\tilde{\delta}$ are further parameters. The $d=1$ Ising model with Glauber kinetics is exactly solvable. In this case the critical temperature is at $T=0$ and the transition is of first order. We recall that $p_T = e^{-4J/kT}$ plays the role of $(T-T_c)/T_c$ in one dimension and, in the vicinity of $T=0$, critical exponents can be defined as powers of p_T. For example, the coherence length ξ satisfies $\xi \propto p_T{}^{-\nu_\perp}$ (see Sec. II). In the presence of a magnetic field B, the magnetization is known exactly. At $T=0$

$$m(T=0,B) = \mathrm{sgn}(B). \tag{53}$$

Moreover, for $\xi \gg 1$ and $B/kT \ll 1$ the exact solution reduces to

$$m \sim 2h\xi;\ \ h = B/k_B T. \tag{54}$$

In scaling form one writes

$$m \sim \xi^{-\beta_s/\nu_\perp} g(h\xi^{\Delta/\nu_\perp}), \tag{55}$$

where Δ is the static magnetic critical exponent. Comparison of Eqs. (54) and (55) results in $\beta_s=0$ and $\Delta=\nu_\perp$. These values are well known for the one–dimensional Ising model. The $\tilde{\delta}=0$, $\Gamma=1$ case is usually referred to as the Glauber-Ising model. The dynamical exponents are (Glauber, 1963; Majumdar *et al.*, 1996)

$$Z_{1d\ Glauber} = 2, \quad \theta_{g,1d\ Glauber} = 1/4. \tag{56}$$

Applying *spin-exchange Kawasaki dynamics*, which conserves the magnetization (model B),

$$w_i = \frac{1}{2\tau}\left[1-\frac{\gamma_2}{2}(s_{i-1}s_i + s_{i+1}si + 2)\right], \tag{57}$$

where $\gamma_2 = \tanh(2J/k_B T)$, we find that the dynamical exponent is different. According to linear response theory (Zwerger, 1981) in one dimension, at the critical point ($T_c=0$) it is

$$Z_{1d\ Kaw} = 5. \tag{58}$$

Note, however, that in the case of a fast quench to $T=0$, coarsening with scaling exponent $1/3$ is reported (Cornell *et al.*, 1991). Hence another dynamic Ising universality class appears with the same static but different dynamical exponents.

Interestingly, while the two-dimensional equilibrium Ising model can be solved, the exact values of the dynamical exponents are not known. Table IV summarizes the known dynamical exponents of the Ising model in $d=1,2,3,4$. The $d=4$ results are mean-field values. In Sec. IV.D.2, I shall discuss another fully nonequilibrium critical point of the $d=1$ Ising model with competing dynamics (the nonequilibrium kinetic Ising model), in which the dynamical exponents break the scaling relation (17) and therefore the magnetization is a non-Markovian process. For $d>3$ there is no non-Markovian effect (hence θ is not independent), but for $d=2,3$ the situation is still not completely clear (Zheng, 1998).

In one dimension the domain walls (kinks) between up and down regions can be considered as particles. The spin-flip dynamics can be mapped onto particle movement,

$$\uparrow\downarrow\downarrow \rightleftharpoons \uparrow\uparrow\downarrow \sim \bullet\circ \rightleftharpoons \circ\bullet \tag{59}$$

or onto the creation or annihilation of neighboring particles,

$$\uparrow\uparrow\uparrow \rightleftharpoons \uparrow\downarrow\uparrow \sim \bigcirc\bigcirc \rightleftharpoons \bullet\bullet. \tag{60}$$

Therefore the $T=0$ Glauber dynamics is equivalent to the diffusion-limited annihilating random walk (ARW) mentioned already in Sec. II. When we map the spin-exchange dynamics in the same way, more complicated particle dynamics emerge, for example,

$$\uparrow\uparrow\downarrow\downarrow \rightleftharpoons \uparrow\downarrow\uparrow\downarrow \sim \bigcirc\bullet\bigcirc \rightleftharpoons \bullet\bullet\bullet. \tag{61}$$

One particle may give birth to two others or three particles may coalesce to one. Therefore these models are equivalent to branching and annihilating random walks, as will be discussed in Sec. IV.D.1.

3. Competing dynamics added to spin flip

Competing dynamics in general break the detailed balance symmetry (49) and cause the kinetic Ising model to relax to a nonequilibrium steady state (if one exists). Generally these models become unsolvable (for an overview, see Rácz, 1996). It was argued by Grinstein *et al.* (1985) that stochastic spin-flip models with two states per site and updating rules of a short-range nature with Z_2 symmetry should belong to the (kinetic) Ising model universality class. This argument is based on the stability of the dynamic Ising fixed point in $d=4-\epsilon$ dimensions with respect to perturbations that retain both spin inversion and lattice symmetries. The hypothesis of Grinstein *et al.* has received extensive confirmation from Monte Carlo simulations (Blote, Heringa, and Zia, 1990; Oliveira *et al.*, 1993; Santos and Teixeira, 1995; Tamayo *et al.*, 1995; Castro *et al.*, 1998) as well as from analytic calculations (Marques, 1989, 1990; Tomé, de Oliveira, and Santos, 1991). The models investigated include Ising models with a competition of two [or three (Tamayo *et al.*, 1995)] Glauber-like rates at different temperatures (Gonzaléz-Miranda *et al.*, 1987; Marques, 1989; Blote *et al.*, 1990; Tomé, de Oliveira, and Santos, 1991), or a combination of spin-flip and spin-exchange dynamics (Garrido *et al.*, 1989), majority vote models (Santos and Teixeira, 1995; Castro *et al.*, 1998), and other types of transition rules with the restrictions mentioned above (Oliveira *et al.*, 1993).

Note that in all of the above cases the ordered state is fluctuating and nonabsorbing. By relaxation processes this allows fluctuations in the bulk of a domain. It is also possible (and in one dimension it is the only choice) to generate nonequilibrium two-state spin models with short-ranged interactions in which the ordered states are frozen (absorbing); hence by relaxation to the steady state, fluctuations occur at the boundaries only. In this case a non-Ising universality class appears, which is called the voter model universality class (see Sec. IV.C).

In two dimensions *general Z_2 symmetric update rules* were investigated by Oliveira *et al.* (1993), Achahbar *et al.* (1996), Drouffe and Godreche (1999), and Dornic *et al.* (2001). These rules are summarized below, following Drouffe and Godreche (1999). Let us consider a two-dimensional lattice of spins $s_i=\pm 1$, evolving with the following dynamical rule. At each evolution step, the spin to be updated flips according to the heat-bath rule: the probability that the spin s_i takes the value +1 is $P(s_i=1)=p(h_i)$, where the local field h_i is the sum over neighboring sites $\Sigma_j s_j$ and

$$p(h)=\frac{1}{2}\{1+\tanh[\beta(h)h]\}. \tag{62}$$

The functions $p(h)$ and $\beta(h)$ are defined over integral values of h. For a square lattice, h takes the values 4, 2, 0, -2, -4. We require that $p(-h)=1-p(h)$, in order to keep up-down symmetry, hence $\beta(-h)=\beta(h)$ and this fixes $p(0)=1/2$. The dynamics therefore depend on two parameters,

$$p_1=p(2), \quad p_2=p(4), \tag{63}$$

or equivalently on two effective temperatures,

$$T_1=\frac{1}{\beta(2)}, \quad T_2=\frac{1}{\beta(4)}. \tag{64}$$

Defining the coordinate system

$$t_1=\tanh\frac{2}{T_1}, \quad t_2=\tanh\frac{2}{T_2} \tag{65}$$

with $0\le t_1,t_2\le 1$ this yields

$$p_1=\frac{1}{2}(1+t_1), \quad p_2=\frac{1}{2}\left(1+\frac{2t_2}{1+t_2^2}\right), \tag{66}$$

with $1/2\le p_1,p_2\le 1$. One can call T_1 and T_2 the temperatures associated with *interfacial noise* and *bulk noise*, respectively. Each point in the parameter plane (p_1,p_2), or alternatively in the temperature plane (t_1,t_2), corresponds to a particular model. The class of models thus defined comprises as special cases the Ising model, the voter and antivoter models (Ligget, 1985), and the majority vote (Ligget, 1985; Castro *et al.*, 1998) model (see Fig. 1). The $p_2=1$ line corresponds to models with no bulk noise ($T_2=0$), hence the dynamics is driven only by interfacial noise, defined above. The $p_1=1$ line corresponds to models with no interfacial noise ($T_1=0$), hence the dynamics is driven only by bulk noise. In both cases effects due to the curvature of the interfaces are always present. For these last models, the local spin aligns in the direction of the majority of its neighbors with probability one, if the local field is $h=2$, i.e., if there is no consensus amongst the neighbors. If there is consensus amongst them, i.e., if $h=4$, the local spin aligns with its neighbors with a probability $p_2<1$.

Simulations (Oliveira *et al.*, 1993) revealed that the transition line between the low- and high-temperature regions is Ising-type except for the end point ($p_1=1,p_2=0.75$), which is first order and corresponds to the voter model class. The local persistence exponent was also found to be constant along the line $\theta_l\sim 0.22$ (Drouffe and Godreche, 1999) in agreement with that of the A model (see Table IV) except for the voter model point. The dynamics of this class of models may be described formally in terms of reaction-diffusion processes for a set of coalescing, annihilating, and branching random

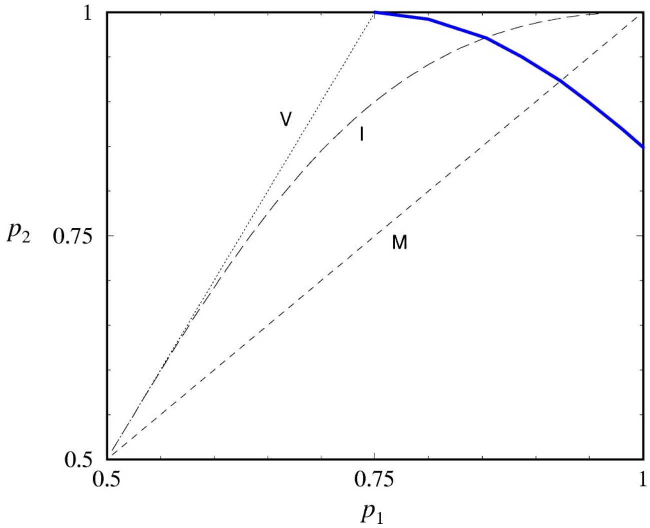

FIG. 1. (Color in online edition) Phase diagram of two-dimensional, Z_2-symmetric nonequilibrium spin models. Dashed lines correspond to the noisy voter model (V), the Ising model (I), and the majority vote model (M). The low-temperature phase is located in the upper right corner, above the transition line (solid line). From Drouffe and Godreche, 1999.

walkers (Drouffe and Godreche, 1999). There are simulation results for other models exhibiting absorbing ordered states, indicating voter model critical behavior (Hinrichsen, 1997; Lipowski and Droz, 2002a).

It is important to note that nontrivial, nonequilibrium phase transitions may occur even in one dimension if spin exchange is added to spin-flip dynamics. The details will be discussed in Sec. IV.D.2.

4. Competing dynamics added to spin exchange

As mentioned in Sec. III, model-B systems are more sensitive to competing dynamics. Local and anisotropic order-parameter-conserving processes generate critical behavior that coincides with that of the kinetic dipolar-interaction Ising model. In two dimensions both simulations and field theory (Praestgaard *et al.*, 1994, 2000) predict the critical exponents. The critical dimension is $d_c=3$. It is shown that the Langevin equation (and therefore the critical behavior) of an anisotropic diffusive system coincides with that of a randomly driven lattice-gas system as well (see Table V). Other systems in this universality class are the two-temperature model (Garrido *et al.*, 1990), the ALGA model (Binder, 1981), and the infinitely fast driven lattice-gas model (Achahbar *et al.*, 2001). In the randomly driven lattice-gas model particle current does not occur but an anisotropy can be found.

TABLE V. Critical exponents of the $d=2$ randomly driven lattice gas.

β	γ	η_a	$\nu_\perp$
0.33(2)	1.16(6)	0.13(4)	0.62(3)

Therefore it was argued (Achahbar *et al.*, 2001) that the particle current is not a relevant feature for this class. This argument offers an explanation for why some sets of simulations of driven lattice systems (Vallés and Marro, 1987) lead to different critical behavior than that of the canonical coarse-grained representative of this class, in which an explicit particle current $j\hat{\mathbf{x}}$ is added to the continuous, model-B Ising-model Hamiltonian (Marro and Dickman, 1999):

$$\frac{\partial\phi(\mathbf{r},t)}{\partial t} = -\nabla\left[\eta\frac{\delta H}{\delta\phi} + j\hat{\mathbf{x}}\right] + \nabla\zeta, \qquad (67)$$

where η is a parameter and ζ is the Gaussian noise. In this model one obtains mean-field exponents for $2\geqslant d\geqslant 5$ (with weak logarithmic corrections at $d=2$) with $\beta=1/2$ exactly. To resolve contradictions between the simulation results of Vallés and Marro (1987) and Leung (1991), Zia *et al.* (2000) suggested the possibility that another, extraordinary, "stringy" ordered phase might exist in Ising-type driven lattice gases, which might be stable in *square* systems.

5. Long-range interactions and correlations

Universal behavior is due to the fact that, at criticality, long-range correlations are generated that make the details of short-ranged interactions irrelevant. However, one can also investigate scaling behavior in systems with long-range interactions or with dynamically generated long-range correlations. If the Glauber Ising model (with nonconserving dynamics) is changed to a nonequilibrium model in such a way that one couples nonlocal dynamics (Droz *et al.*, 1990) to it, long-range isotropic interactions are generated and mean-field critical behavior emerges. For example, if the nonlocal dynamics is a random Lévy flight with a spin-exchange probability distribution

$$P(r) \propto \frac{1}{r^{d+\sigma}}, \qquad (68)$$

then effective long-range interactions of the form $V_{\rm eff}\propto r^{-d-\sigma}$ are generated and the critical exponents change continuously as a function of σ and d (Bergersen and Rácz, 1991). Similar conclusions for other nonequilibrium classes will be discussed later (Sec. IV.A.6).

The effect of power-law correlated initial conditions $\langle\phi(0)\phi(r)\rangle\sim r^{-(d-\sigma)}$ in the case of a quench to the ordered phase in systems with nonconserved order parameter was investigated by Bray *et al.* (1991). An important example is the (2+1)-dimensional Glauber-Ising model quenched to zero temperature. It was observed that long-range correlations are relevant if σ exceeds a critical value σ_c. Furthermore, it was shown that the relevant regime is characterized by a continuously changing exponent in the autocorrelation function, $A(t)=[\phi(r,t)\phi(r,0)]\sim t^{-(d-\sigma)/4}$, whereas the usual short-range scaling exponents could be recovered below the threshold. These features are in agreement with simulations of the two-dimensional Ising model quenched from $T=T_c$ to $T=0$.

6. Damage-spreading behavior

The high-temperature phase of the Ising model is chaotic. When T is lowered a nonchaotic phase may emerge at T_d with different types of transitions depending on the dynamics. The dynamics-dependent damage-spreading (DS) critical behavior in different Ising models is in agreement with a conjecture by Grassberger (1995c). Dynamical simulations with the heat-bath algorithm in two and three dimensions (Gropengiesser, 1994; Grassberger, 1995a; Wang and Suzuki, 1996) resulted in $T_d=T_c$ with a DS dynamical exponent coinciding with that of the Z's of the replicas. In this case the DS transition picks up the Ising class universality of its replicas. With Glauber dynamics in two dimensions $T_d<T_c$, and directed-percolation class DS exponents were found (Grassberger, 1995b). With Kawasaki dynamics in two dimensions, on the other hand, the damage always spreads (Vojta, 1998). With Swendsen-Wang dynamics in two dimensions $T_d>T_c$, and directed-percolation class DS behavior was observed (Hinrichsen *et al.*, 1998).

In one-dimensional, nonequilibrium Ising models it is possible to design different dynamics showing either a parity-conserving or a directed-percolation-class DS transition as the function of some control parameter. This depends on whether the DS transition coincides or not with the critical point. In the parity-conserving class DS case, damage variables exhibit the dynamics of a two-offspring branching and annihilating random walk (BARW2; see Secs. IV.D.1 and IV.D.2), and Z_2 symmetric absorbing states occur (Hinrichsen and Domany, 1997; Ódor and Menyhárd, 1998).

B. Potts classes

The generalization of the two-state equilibrium Ising model was introduced by Potts (1952); for an overview see Wu (1982). In the q-state Potts model the state variables can take q different values $s_i \in (0,1,2,\ldots,q)$ and the Hamiltonian is a sum of Kronecker delta functions of states over nearest neighbors,

$$H=-J\sum_{\langle i,i'\rangle}\delta(s_i-s_{i'}). \tag{69}$$

This Hamiltonian exhibits a global symmetry described by the permutation group of q elements (S_q). The Ising model is recovered in the $q=2$ case (discussed in Sec. III.A). The q-state Potts model exhibits a disordered high-temperature phase and an ordered low-temperature phase. The transition is first order and mean-field-like for q's above the $q_c(d)$ curve (shown in Fig. 2) and for $q>2$ in high dimensions. The $q=1$ limit can be shown (Fortuin and Kasteleyn, 1972) to be equivalent to the isotropic percolation (see Sec. IV.B.1), which is known to exhibit a continuous phase transition

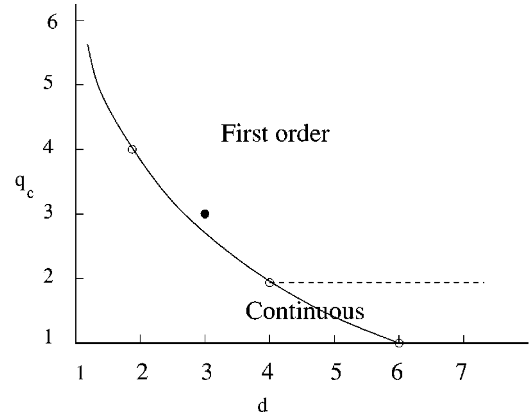

FIG. 2. Schematic plot for $q_c(d)$ (solid line). Open symbols correspond to continuous phase transition, filled symbol to a known first-order transition. Below the dashed line the transition is continuous, too. From Wu, 1982.

with $d_c=6$. The problem of finding the effective resistance between two nodes of a network of linear resistors was solved by Kirchhoff in 1847. Fortuin and Kasteleyn (1972) showed that Kirchoff's solution can be expressed as a $q=0$ limit of the Potts partition function. Further mappings were discovered between the spin glass (Edwards and Anderson, 1975) and the $q=1/2$ Potts model and between the two-dimensional $q=3,4$ cases and vertex models (see Baxter, 1982). From Fig. 2 we can see that, for $q>2$ Potts models, continuous transitions occur in two dimensions only ($q=3,4$). Fortunately these models are exactly solvable (see Baxter, 1982) and exhibit conformal symmetry as well as topological Yang-Baxter invariance. The static exponents in two dimensions are known exactly (Table VI).

1. Correlated percolation at T_c

In two-dimensional models with Z_3 symmetry, the critical point coincides with the percolation of site connected clusters, and the following percolation exponents are reported (Fortunato and Satz, 2001):

$$\beta_p=0.075(14), \quad \gamma_p=1.53(21). \tag{70}$$

In the case of Fortuin-Kasteleyn cluster construction (Fortuin and Kasteleyn, 1972), the percolation exponents of the q-state Potts model at T_c coincide with those of the magnetization of the model.

TABLE VI. Static exponents of the q states Potts model in two dimensions.

Exponent	$q=0$	$q=1$	$q=2$	$q=3$	$q=4$
α_H	$-\infty$	$-2/3$	0(ln)	1/3	2/3
β	1/6	5/36	1/8	1/9	1/12
γ	∞	43/18	7/4	13/9	7/6
$\nu_\perp$	∞	4/3	1	5/6	2/3

TABLE VII. Model-A dynamical exponents of the q-state Potts model in two dimensions.

Exponent	$q=3$	$q=4$
Z	2.198(2)	2.290(3)
λ	0.836(2)	
θ_g	0.350(8)	

2. Dynamical Potts classes

The model-A dynamical exponents of the Potts classes have been determined in two dimensions for $q=3,4$ by short-time Monte Carlo simulations (de Silva *et al.*, 2002; Table VII). The exponents were found to be the same for heat-bath and Metropolis algorithms. For the zero-temperature *local persistence exponent* in one dimension, exact formulas have been determined. For sequential dynamics (Derrida *et al.*, 1995)

$$\theta_{l,s} = -\frac{1}{8} + \frac{2}{\pi^2}\left[\cos^{-1}\left(\frac{2-q}{\sqrt{2}q}\right)\right]^2, \tag{71}$$

while for parallel dynamics $\theta_{l,p}=2\theta_{l,s}$ (Menon and Ray, 2001). In a deterministic coarsening θ_l is again different (see Bray *et al.*, 1994; Gopinathan, 1998). As we can see the dynamical universality class characterized by the growth of the length as $\xi \propto t^{1/2}$ is divided into subclasses, as reflected by the persistence exponent.

As in the Ising model in nonequilibrium systems, symmetry in the Potts model turned out to be a relevant factor for determining the universal behavior of transitions to fluctuating ordered states (Crisanti and Grassberger, 1994; Brunstein and Tomé, 1998; Szabó and Czárán, 2001). On the other hand, in the case of nonequilibrium transitions to absorbing states, a simulation study (Lipowski and Droz, 2002a) suggests a first-order transition for all $q>2$ Potts models in $d>1$ dimensions. The $q=2$, $d=2$ case corresponds to the voter model class (Sec. IV.C) and in one dimension either a parity-conserving class transition ($q=2$) or N-BARW2 class transition ($q=3$) (Sec. V.K) occurs.

The DS transition of a $q=3$,$d=2$ Potts model with heat-bath dynamics was found to belong to the directed-percolation class (Sec. IV.A) because $T_d > T_c$ (da Silva *et al.*, 1997).

3. Long-range interactions

The effect of long-range interaction has also been investigated in the case of the one-dimensional $q=3$ Potts model (Glumac and Uzelac, 1998) with the Hamiltonian

$$H = -\sum_{i<j} \frac{J}{|i-j|^{1+\sigma}} \delta(s_i,s_j). \tag{72}$$

For $\sigma<\sigma_c \sim 0.65$ a crossover from second-order to first-order (mean-field) transition was located by simulations. As in the Ising model, here we can expect to see this crossover if we generate the long-range interactions by the addition of a Lévy-type Kawasaki dynamics.

TABLE VIII. Static exponents of the XY model in three dimensions.

α_H	β	γ	$\nu_\perp$	η_a
−0.011(4)	0.347(1)	1.317(2)	0.670(1)	0.035(2)

C. *XY* model classes

The classical XY model is defined by the Hamiltonian

$$H = -J\sum_{i,i'} \cos(\Theta_i - \Theta_{i'}), \tag{73}$$

with continuous $\Theta_i \in [0,2\pi]$ state variables. This model has a global U(1) symmetry. Alternatively the XY model can be defined as a special $N=2$ case of O(N) symmetric models such that the spin vectors $\mathbf{S}_i$ are two dimensional with absolute value $\mathbf{S}_i^2=1$,

$$H = -J\sum_{\langle i,i'\rangle} \mathbf{S}_i\mathbf{S}_{i'}. \tag{74}$$

In this continuous model in two dimensions no local order parameter can take zero value according to the Mermin-Wagner theorem (Mermin and Wagner, 1996). The appearance of free vortices (which are nonlocal) causes an unusual transition mechanism that implies that most of the thermodynamic quantities do not show power-law singularities. The singular behavior of the correlation length (ξ) and the susceptibility (χ) is described by the forms for $T>T_c$,

$$\xi \propto \exp[C(T-T_c)^{-1/2}], \quad \chi \propto \xi^{2-\eta_a}, \tag{75}$$

where C is a nonuniversal positive constant. Conventional critical exponents cannot be used, but one can define scaling dimensions. At T_c the two-point correlation function has the following long-distance behavior:

$$G(r) \propto r^{-1/4}(\ln r)^{1/8}, \tag{76}$$

implying $\eta_a=1/4$, and in the entire low-temperature phase,

$$G(r) \propto r^{-\eta_a(T)} \tag{77}$$

such that the exponent η_a is a continuous function of the temperature, i.e., the model has a line of critical points starting from T_c and extending to $T=0$. This is the so-called Kosterlitz-Thouless critical behavior (Kosterlitz and Thouless, 1973) and corresponds to the conformal field theory with $c=1$ (Itzykson and Drouffe, 1989). This kind of transition can be experimentally observed in many effectively two-dimensional systems with O(2) symmetry, such as thin films of superfluid helium, as well as in roughening transitions of SOS models at crystal interfaces. In three dimensions the critical exponents of the O(N) ($N=0,1,2,3,4$) symmetric field theory have been determined by perturbative expansions up to seventh-loop order (Guida and Zinn-Justin, 1998). Table VIII summarizes these results for the XY case. For a more detailed treatment see Pelissetto and Vicari (2002).

TABLE IX. Model-A dynamical exponents of the XY model in two dimensions.

Z	λ	η
2.04(1)	0.730(1)	0.250(2)

Exponents with model-A dynamics in two dimensions have been determined by short-time simulations, and logarithmic corrections to scaling were found (Ying *et al.*, 2001). Table IX summarizes the known dynamical exponents

1. Long-range correlations

Similarly to the Ising model, Bassler and Rácz (1995) studied the validity of the Mermin-Wagner theorem by transforming the two-dimensional XY model to a nonequilibrium one using two-temperature, model-A dynamics. They found that the Mermin-Wagner theorem does not apply for this case, since effective long-range interactions are generated by the local nonequilibrium dynamics. The universality class of the phase transition of the model coincides with that of the two-temperature driven Ising model (see Table V).

2. Self-propelled particles

Another XY-like nonequilibrium model that exhibits an ordered state for $d \leqslant 2$ dimensions is motivated by the description of the "flocking" behavior among living things, such as birds, slime molds, and bacteria. In the simplest version of the self-propelled particle model (Vicsek *et al.*, 1995), each particle's velocity is set to a fixed magnitude v_0. Interaction with the neighboring particles changes only the direction of motion: the particles tend to align their orientation to the local average velocity. In one dimension the lattice dynamics are defined as

$$x_i(t+1) = x_i(t) + v_0 u_i(t),$$

$$u_i(t+1) = G[\langle u(t)\rangle_i] + \xi_i, \tag{78}$$

where the particles are characterized by their coordinate x_i and dimensionless velocity u_i. The function G incorporates both the propulsion and friction forces which set the average velocity to a prescribed value v_0: $G(u) > u$ for $u < 1$ and $G(u) < u$ for $u > 1$. The distribution function $P(x=\xi)$ of the noise ξ_i is uniform in the interval $[-\eta/2, \eta/2]$. When v_0 is kept constant, the adjustable control parameters of the model are the average density of the particles, ρ, and the noise amplitude η. The order parameter is the average velocity $\phi \equiv \langle u\rangle$, which vanishes as

$$\phi(\eta,\rho) \sim \begin{cases} \left(\dfrac{\eta_c(\rho) - \eta}{\eta_c(\rho)}\right)^{\beta} & \text{for } \eta < \eta_c(\rho) \\ 0 & \text{for } \eta > \eta_c(\rho), \end{cases} \tag{79}$$

at a critical $\eta_c(\rho)$ value.

This model is similar to the XY model of classical magnetic spins because the velocity of the particles, like the local spin of the XY model, has fixed length and continuous rotational U(1) symmetry. In the $v_0 = 0$ and low-noise limit the model reduces *exactly* to the Monte Carlo dynamics of the XY model.

A field theory that included nonequilibrium effects in a self-consistent way was proposed by Tu and Toner (1995). Their model is different from the XY model for $d < 4$. The essential difference between the self-propelled-particle model and the equilibrium XY model is that at different times, the "neighbors" of one particular "bird" will be different depending on the velocity field itself. Therefore two originally distant birds can interact with each other at some later time. Tu and Toner found a critical dimension $d_c = 4$, below which linearized hydrodynamics breaks down, but owing to Galilean invariance they could obtain exact scaling exponents in $d = 2$. For the dynamical exponent they got $Z = 6/5$. Numerical simulations (Vicsek *et al.*, 1995; Czirók *et al.*, 1997) indeed found a long-range-ordered state with a continuous transition characterized by $\beta = 0.42(3)$ in two dimensions.

In one dimension the field theory and simulations (Czirók *et al.*, 1999) have provided evidence for a continuous phase transition with $\beta = 0.60(5)$, which is different from the mean-field value $1/2$ (Stanley, 1971).

D. O(N) symmetric model classes

As already mentioned in the previous section the O(N) symmetric models are defined on spin vectors $\mathbf{S}_i$ of unit length $\mathbf{S}_i^2 = 1$ with the Hamiltonian

$$H = -J \sum_{\langle i,i'\rangle} \mathbf{S}_i \mathbf{S}_i'. \tag{80}$$

The best known of these models is the classical Heisenberg model, which corresponds to $N = 3$. This is the simplest model for an isotropic ferromagnet. The $N = 4$ case corresponds to the Higgs sector of the Standard Model at finite temperature. The $N = 0$ case is related to polymers, and the $N = 1$ and $N = 2$ cases are the Ising and XY models, respectively. The critical dimension is $d_c = 4$ and, by the Mermin-Wagner theorem, we cannot find a finite-temperature phase transition in the short-range equilibrium models for $N > 2$ below $d = 3$. The static critical exponents have been determined by $\epsilon = 4 - d$ expansions up to five-loop order (Gorishny *et al.*, 1984), by exact RG methods (see Berges *et al.*, 2002 and references therein), by simulations (Hasenbuch, 2001) and by series expansions in three dimensions (see Guida and Zinn-Justin, 1998, and references therein). In Table X, I show the latest estimates from Guida and Zinn-Justin (1998) in three dimensions for $N = 0,3,4$. The $N \to \infty$ limit is the exactly solvable spherical model (Berlin and Kac, 1952; Stanley, 1968). For a detailed discussion of the static critical behavior of the O(N) models see Pelissetto and Vicari (2002).

TABLE X. Static exponents of the O(N) model in three dimensions.

N	α_H	β	γ	$\nu_\perp$	η_a
0	0.235(3)	0.3024(1)	1.597(2)	0.588(1)	0.028(2)
3	−0.12(1)	0.366(2)	1.395(5)	0.707(3)	0.035(2)
4	−0.22(2)	0.383(4)	1.45(1)	0.741(6)	0.035(4)

The dynamical exponents for model A are known exactly for the ($N\to\infty$) spherical model (Janssen *et al.*, 1989; Majumdar *et al.*, 1996; see Table XI). For other cases $\epsilon=4-d$ expansions up to two-loop order exist (Majumdar *et al.*, 1996; Oerding *et al.*, 1997). For a discussion about the combination of different dynamics see the general introduction, Sec. III, and Täuber, Santos, and Rácz (1999).

In three dimensions for O(2), O(3), and O(4) symmetric models, the Fortuin-Kasteleyn cluster construction (Fortuin and Kasteleyn, 1972) results in percolation points and percolation exponents that coincide with the corresponding T_c's and magnetization exponent values (Blanchard *et al.*, 2000).

IV. GENUINE, BASIC NONEQUILIBRIUM CLASSES

In this section I introduce "genuine nonequilibrium" universality classes that do not occur in dynamical generalizations of equilibrium systems. Naturally in these models there is no Hermitian Hamiltonian, and they are defined by transition rates not satisfying the detailed balance condition (49). They can be described by a master equation and by the deduced stochastic action or Langevin equation if it exists. The best-known cases are reaction-diffusion systems with order-disorder transitions in which the ordered state may exhibit only small fluctuations, hence they trap or absorb a system falling into them. Examples occur in models of population (Albano, 1994), epidemics (Mollison, 1977; Ligget, 1985), catalysis (Ziff *et al.*, 1986), or enzyme biology (Berry, 2003). There are also other nonequilibrium phase transitions, for example, in lattice gases with currents (Evans *et al.*, 1995, 1998; Kolomeisky *et al.*, 1998; Evans, 2000) or in traffic models (Chowdhury *et al.*, 2000), but in these systems the critical universality classes have not yet been explored.

Phase transitions in such models may occur in low dimensions, in contrast with equilibrium models (Marro and Dickman, 1999). As was already shown in Sec. III.A.2, reaction-diffusion particle systems may be mapped onto spin-flip systems, stochastic cellular automata (Chopard and Droz, 1998), or interface growth models (see Sec. VI). The mapping, however, may lead to nonlocal systems that have no physical relevance. The universality classes of the simple models presented in this section constitute the fundamental building blocks of more complex systems.

TABLE XI. Model-A dynamical exponents of the O(N) model in three dimensions.

N	Z	λ	θ_g
∞	2	5/2	1/4
3	2.032(4)	2.789(6)	0.38

For a long time, phase transitions with completely frozen absorbing states were investigated. A few universality classes of this kind were known (Grassberger, 1996; Hinrichsen, 2000a), of which the most prominent and the first to be discovered was that of *directed percolation* (DP; Kinzel, 1983). An early hypothesis (Janssen, 1981; Grassberger, 1982a; Grinstein *et al.*, 1989) has been confirmed by all models up to now. This conjecture, known as the *DP hypothesis*, claims that *in one-component systems exhibiting continuous phase transitions to a single absorbing state (without extra symmetry, inhomogeneity, or disorder) short-ranged interactions can generate DP-class transitions only*. Despite the robustness of this class, experimental observation is still lacking (Grassberger, 1996; Hinrichsen, 2000c), probably owing to the sensitivity to disorder that cannot be avoided in real materials.

A major problem with the analyses of these models is that they are usually far from the critical dimension, and critical fluctuations prohibit mean-field-like behavior. A further complication is that bosonic field-theoretical methods cannot describe the particle exclusion that may obviously happen in $d=1$. The success of the application of bosonic field theory in many cases is the consequence of the asymptotically low density of particles near the critical point. However, in multicomponent systems, where the exchange between different types is nontrivial, bosonic field-theoretical descriptions may fail. In the case of binary production models (Sec. V.F) the bosonic renormalization group predicts diverging density in the active phase, in contrast to the lattice-model version of hard-core particles (Ódor, 2000, 2001a; Carlon *et al.*, 2001; Hinrichsen, 2001b, 2001c; Park *et al.*, 2001). Fermionic field theories, on the other hand, have the disadvantage that they are nonlocal, hence results exist for very simple reaction-diffusion systems only (Brunel *et al.*, 2000; Park *et al.*, 2000; Wijland, 2001). Other techniques like independent interval approximation (Krapivsky and Ben-Naim, 1997), the empty-interval method (ben Avraham *et al.*, 1990), series expansion (Essam *et al.*, 1996), or density matrix renormalization (DMRG) are currently under development.

The universal scaling-law behavior in these models is described by the critical exponents in the neighborhood of a steady state, hence the generalization of dynamical exponents (like Z, θ, λ, etc.) introduced for "out of equilibrium classes" (Sec. III). Besides these exponents, there are genuinely nonequilibrium dynamical exponents, as well, to characterize spreading behavior, defined in Sec. I.D. For each class I discuss the damage-spreading transitions, the effects of different boundary conditions, disorder, and long-range correlations generated by anomalous diffusion or by special initial states.

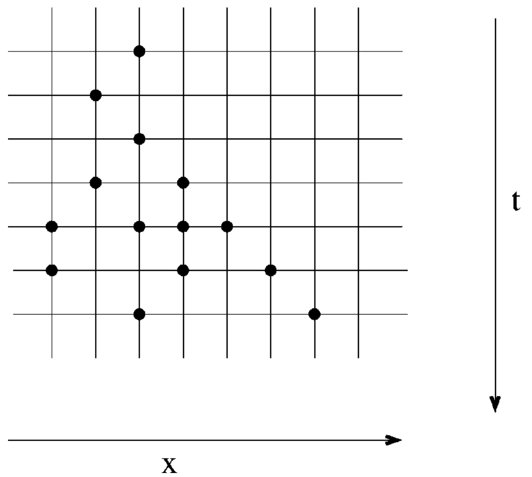

FIG. 3. Directed site percolation in $d=1+1$ dimensions.

A. Directed-percolation classes

The directed percolation introduced by Broadbent and Hammersley (1957) is an anisotropic percolation with a preferred direction t. This means that this problem should be $d \geqslant 2$ dimensional. If there is an object (bond, site, etc.) at $(x_i, y_j, \ldots, t_k)$, it must have a nearest-neighboring object at t_{k-1} unless $t_k=0$ (see Fig. 3). If we consider the preferred direction as the time, we recognize a spreading process of an agent A that cannot have a spontaneous source: $\varnothing \nrightarrow A$. This results in the possibility of a completely frozen, so-called *absorbing state* from which the system cannot escape if it has fallen into it. As a consequence these kinds of models may have phase transitions in $d=1$ spatial dimension already. By increasing the branching probability p of the agent we can have a phase transition between the absorbing state and an active steady state with finite density of A's. If the transition is continuous, it is very likely that it belongs to a robust universality class, the directed-percolation (DP) class.

For a long time all models of such absorbing phase transitions were found to belong to the DP class, and the DP hypothesis was advanced (see Sec. IV above). This hypothesis has been confirmed by all models up to now. Moreover, DP class exponents were discovered in some systems with multiple absorbing states. For example, in systems with infinitely many frozen absorbing states (Jensen, 1993a; Jensen and Dickman, 1993a; Mendes *et al.*, 1994) the static exponents were found to coincide with those of directed percolation. Furthermore DP behavior was reported by models without any special symmetry of the absorbing states (Park and Park, 1995; Menyhárd and Ódor, 1996; Ódor and Menyhárd, 1998). So, although the necessary conditions for DP behavior seem to be confirmed, the determination of sufficient conditions is an open problem. There are many introductory works available now on directed percolation (Kinzel, 1983; Grassberger, 1996; Marro and Dickman, 1999; Hinrichsen, 2000a). Therefore I shall not go very deeply into a discussion of the details of various representations.

In the reaction-diffusion language, directed percolation is built up from the following processes:

$$A \xrightarrow{\gamma} \varnothing, \quad A\varnothing \overset{D}{\leftrightarrow} \varnothing A, \quad A \xrightarrow{\sigma} 2A, \quad 2A \xrightarrow{\gamma} A. \tag{81}$$

The mean-field equation for the coarse-grained particle density $\rho(t)$ is

$$\frac{d\rho}{dt} = (\sigma - \gamma)\rho - (\lambda + \sigma)\rho^2. \tag{82}$$

This has the stationary stable solution

$$\rho(\infty) = \begin{cases} \dfrac{\sigma-\gamma}{\lambda+\sigma} & \text{for } \sigma > \gamma \\ 0 & \text{for } \sigma \leqslant \gamma \end{cases} \tag{83}$$

exhibiting a continuous transition at $\sigma=\gamma$. A small variation of σ or γ near the critical point implies a linear change of ρ. Therefore the order-parameter exponent in the mean-field approximation is $\beta=1$. Near the critical point the $O(\rho)$ term is the dominant one, hence the density approaches the stationary value exponentially. For $\sigma=\gamma$ the remaining $O(\rho^2)$ term causes a power-law decay: $\rho \propto t^{-1}$ indicating $\alpha=1$. To get information about other scaling exponents we have to take into account the diffusion term $D\nabla^2$ describing local density fluctuations. Using rescaling invariance two more independent exponents can be determined: $\nu_\perp=1/2$ and $Z=2$ if $d \geqslant 4$. Therefore the upper critical dimension of directed percolation is $d_c=4$.

Below the critical dimension, RG analysis of the Langevin equation

$$\frac{\partial \rho(x,t)}{\partial t} = D\nabla^2 \rho(x,t) + (\sigma-\gamma)\rho(x,t) - (\lambda+\sigma)\rho^2(x,t) + \sqrt{\rho(x,t)}\,\eta(x,t) \tag{84}$$

is necessary (Janssen, 1981). Here $\eta(x,t)$ is the Gaussian noise field, defined by the correlations

$$\langle \eta(x,t) \rangle = 0, \tag{85}$$

$$\langle \eta(x,t)\eta(x',t') \rangle = \Gamma \delta^d(x-x')\delta(t-t'). \tag{86}$$

The noise term is proportional to $\sqrt{\rho(x,t)}$, ensuring that in the absorbing state $[\rho(x,t)=0]$ it vanishes. The square-root behavior stems from the definition of $\rho(x,t)$ as a coarse-grained density of active sites averaged over some mesoscopic box size. Note that DP universality occurs in many other processes, such as odd-offspring, branching and annihilating random walks (BARWo; see Sec. IV.A.3), and in models described by field theory with higher-order terms like $\rho^3(x,t)$ or $\nabla^4\rho(x,t)$, which are irrelevant under the RG transformation. This stochastic process can, through standard techniques (Janssen, 1976), be transformed into a Lagrangian formulation with the action

TABLE XII. Estimates for the critical exponents of directed percolation. One-dimensional data are from Jensen (1999a); two-dimensional data are from Voigt and Ziff (1997); three-dimensional data are from Jensen (1992); four-dimensional-ϵ data are from Bronzan and Dash (1974) and Janssen (1981).

Critical exponent	$d=1$	$d=2$	$d=3$	$d=4-\epsilon$
$\beta=\beta'$	0.276486(8)	0.584(4)	0.81(1)	$1-\epsilon/6-0.01128\epsilon^2$
$\nu_\perp$	1.096854(4)	0.734(4)	0.581(5)	$1/2+\epsilon/16+0.02110\epsilon^2$
$\nu_\parallel$	1.733847(6)	1.295(6)	1.105(5)	$1+\epsilon/12+0.02238\epsilon^2$
$Z=2/z$	1.580745(10)	1.76(3)	1.90(1)	$2-\epsilon/12-0.02921\epsilon^2$
$\delta=\alpha$	0.159464(6)	0.451	0.73	$1-\epsilon/4-0.01283\epsilon^2$
η	0.313686(8)	0.230	0.12	$\epsilon/12+0.03751\epsilon^2$
γ_p	2.277730(5)	1.60	1.25	$1+\epsilon/6+0.06683\epsilon^2$

$$S=\int d^dxdt\left[\frac{D}{2}\psi^2\phi+\psi(\partial_t\phi-\nabla^2\phi-r\phi+u\phi^2)\right], \quad (87)$$

where ϕ is the density field and ψ is the response field (appearing in response functions), and the action is invariant under the following time-reversal symmetry:

$$\phi(x,t)\to-\psi(x,-t),\quad \psi(x,t)\to-\phi(x,-t). \quad (88)$$

This symmetry yields (Grassberger and de la Torre, 1979; Muñoz *et al.*, 1997) the scaling relations

$$\beta=\beta' \quad (89)$$

$$4\delta+2\eta=dz. \quad (90)$$

This field theory was found to be equivalent (Cardy and Sugar, 1980) to the Reggeon field theory (Abarbanel *et al.*, 1975; Brower *et al.*, 1978), a model of elementary particles scattered at high energies and low-momentum transfers.

Perturbative $\epsilon=4-d$ renormalization-group analysis (Bronzan and Dash, 1974; Janssen, 1981) up to two-loop order resulted in estimates for the critical exponents shown in Table XII. The best results are obtained by approximative techniques for the DP-like improved mean-field approximation (Ben-Naim and Krapivsky, 1994), the coherent anomaly method (Ódor, 1995), Monte Carlo simulations (Grassberger and de la Torre, 1979; Grassberger, 1989a, 1992a; Dickman and Jensen, 1991), series expansions (De'Bell and Essam, 1983; Essam *et al.*, 1988; Jensen and Dickman, 1993b, 1993c; Jensen and Guttmann, 1995, 1996; Jensen, 1996a, 1999a), DMRG (Hieida, 1998; Carlon *et al.*, 1999), and numerical integration of Eq. (84) (Dickman, 1994).

The *local persistence probability* may be defined as the probability $[p_l(t)]$ that a particular site never becomes active up to time t. Numerical simulations (Hinrichsen and Koduvely, 1998) for this in the (1+1)-dimensional Domany-Kinzel stochastic cellular automaton (see Sec. IV.A.2) found a power-law behavior with the exponent

$$\theta_l=1.50(1). \quad (91)$$

The *global persistence probability* is defined here as the probability $[p_g(t)]$ that the deviation of the global density from its mean value does not change its sign up to time t. The simulations of Hinrichsen and Koduvely (1998) in 1+1 dimensions claim $\theta_g\geqslant\theta_l$. This agrees with the field-theoretical $\epsilon=4-d$ expansions of Oerding and van Wijland (1998), who predict $\theta_g=2$ for $d\geqslant4$ and for $d<4$,

$$\theta_g=2-\frac{5\epsilon}{24}+O(\epsilon^2). \quad (92)$$

The crossover from isotropic to directed percolation was investigated by Frey *et al.* (1993), using the perturbative RG approach up to one-loop order. They found that while for $d>6$ the percolation is isotropic, for $d>5$ the directed Gaussian fixed point is stable. For $d<5$ the asymptotic behavior is governed by the directed-percolation fixed point. On the other hand, in $d=2$, exact calculations and simulations (Fröjdh and den Nijs, 1997) found that the isotropic percolation is stable with respect to directed percolation if we control the crossover by a spontaneous particle birth parameter. It is still an open question what happens for $2<d<5$. Crossovers to mean-field behavior generated by long-range interactions (Sec. IV.A.7) and crossovers to compact directed percolation (Sec. IV.C) will be discussed later.

It was conjectured (Deloubrière and van Wijland, 2002) that in one-dimensional "fermionic" (single site occupancy) and bosonic (multiple site occupancy) models may exhibit different critical behavior. An attempt at a fermionic field-theoretical treatment of directed percolation in (1+1) dimension was made by Brunel *et al.* (2000; see also Wijland, 2001). This ran into severe convergence problems and did not result in precise quantitative estimates for the critical exponents. Although the bosonic field theory is expected to describe the fermionic case, owing to the asymptotically low density at criticality it has never been proven rigorously. Since analytic results are known only for bosonic field theory, which gives rather inaccurate critical exponent estimates, Ódor and Menyhárd (2002) performed simulations of a BARW reaction-diffusion process [see Eq. (101)] with unrestricted site occupancy to investigate the density $[\rho(t)]$ decay of a DP process from a random ini-

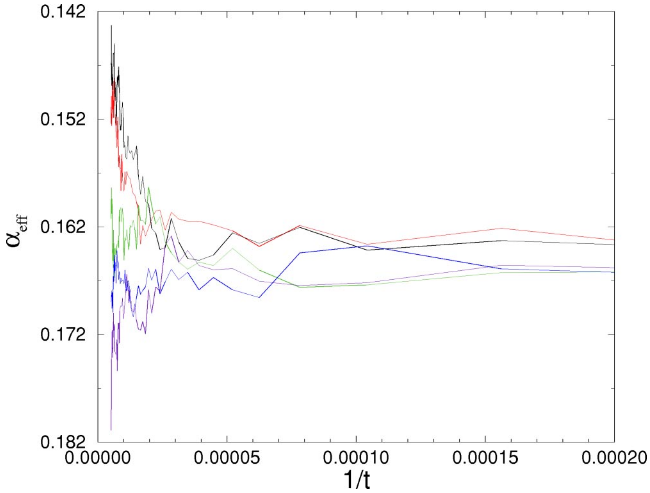

FIG. 4. (Color in online edition) Local slopes of the density decay in a Bosonic BARW1 model (Sec. IV.A.3). Different curves correspond to $\lambda=0.12883, 0.12882, 0.12881, 0.1288, 0.12879$ (from bottom to top). From Ódor and Menyhárd, 2002.

tial state. Figure 4 shows the local slopes of the density decay defined as

$$\alpha_{\rm eff} = -\frac{d \ln \rho(t)}{d \ln t} \tag{93}$$

around the critical point for several annihilation rates (λ). The critical point is estimated at $\lambda_c=0.128\,82(1)$ (corresponding to a straight line), with the extrapolated decay exponent $\alpha=0.165(5)$, which agrees well with fermionic model simulation and series-expansion results of $\lambda_c=0.1595(1)$ (De'Bell and Essam, 1983).

Note that in site-restricted models the processes

$$A \xrightarrow{\gamma} \emptyset\,, \quad A\emptyset \overset{D}{\leftrightarrow} \emptyset A\,, \quad A \xrightarrow{\sigma} 2A \tag{94}$$

generate a DP-class phase transition, while in the bosonic version the $2A \xrightarrow{\lambda} A$ process is also necessary to ensure an active steady state (without it the density diverges for $\sigma > \gamma$).

Models exhibiting DP transitions have been reviewed in great detail by Marro and Dickman (1999) and Hinrichsen (2000a). In the next subsections I discuss only three important examples.

1. The contact process

The contact process is one of the earliest and simplest lattice models for directed percolation, with asynchronous updates introduced by Harris (1974b) and Grassberger and de la Torre (1979) to model epidemic spreading without immunization. Its dynamics is defined by nearest-neighbor processes that occur spontaneously due to specific rates (rather than probabilities). In numerical simulations, models of this type are usually realized by random sequential updates. In one dimension this means that a pair of sites $\{s_i, s_{i+1}\}$ is chosen at random and an update is attempted according to specific

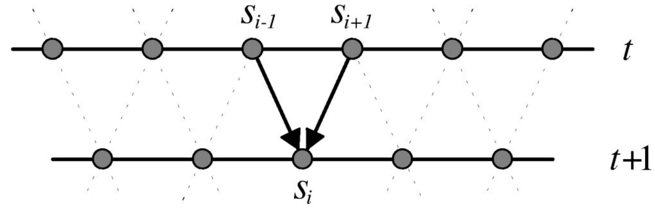

FIG. 5. Schematic of the update rule for the Domany-Kinzel model.

transition rates $w(s_{i,t+dt}, s_{i+1,t+dt} | s_{i,t}, s_{i+1,t})$. Each attempt to update a pair of sites increases the time t by $dt = 1/N$, where N is the total number of sites. One time step (sweep) therefore consists of N such attempts. The contact process is defined by the rates

$$w(A,I|A,A) = w(I,A|A,A) = \mu, \tag{95}$$

$$w(I,I|A,I) = w(I,I|I,A) = \lambda, \tag{96}$$

$$w(A,A|A,I) = w(A,A|I,A) = 1, \tag{97}$$

where $\lambda > 0$ and $\mu > 0$ are two parameters (all other rates are zero). Equation (95) describes the creation of inactive (I) spots within active (A) islands. Equations (96) and (97) describe the shrinkage and growth of active islands. In order to fix the time scale, we chose the rate in Eq. (97) to be equal to one. The active phase is restricted to the region $\lambda < 1$, where active islands are likely to grow. In one dimension, series expansions and numerical simulations have determined the critical point and critical exponents precisely.[1] In two dimensions, the order-parameter moments and the cumulant ratios were determined by Dickman and de Silva (1998a, 1998b).

2. Directed-percolation-class stochastic cellular automata

Cellular automata as the simplest systems exhibiting synchronous dynamics have been extensively studied (Wolfram, 1983). When the update rules are made probabilistic, phase transitions as a function of some control parameter may emerge. There are many stochastic cellular automata (SCA) that exhibit DP transitions (Boccara and Roger, 1993), perhaps the first and simplest being the (1+1)-dimensional Domany-Kinzel model (Domany and Kinzel, 1984). In this model the state at a given time t is specified by binary variables $\{s_i\}$, which can have the values A (active) and I (inactive). At odd times the odd-indexed states are updated, whereas at even times the rest of the sites are updated according to specific conditional probabilities. This model defines a cellular automaton with parallel updates (discrete time evolution) acting on two independent triangular sublattices (Fig. 5). The conditional probabilities in the

[1] See, for example, De'Bell and Essam, 1983; Essam *et al.*, 1988; Dickman and Jensen, 1991; Jensen and Dickman, 1993b, 1993c; Jensen and Guttmann, 1995, 1996; Jensen, 1996a; Dickman and de Silva, 1998a, 1998b.

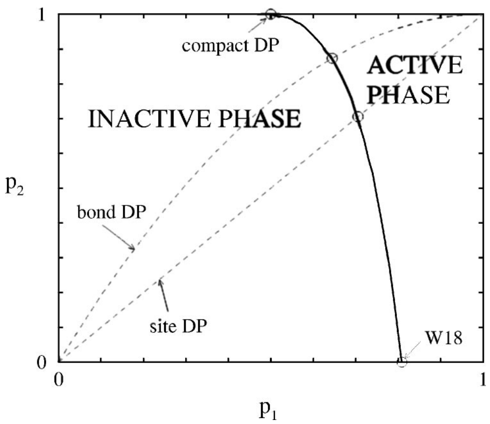

FIG. 6. Phase diagram of the one-dimensional Domany-Kinzel stochastic cellular automation. From Hinrichsen, 1997.

Domany-Kinzel model $P(s_{i,t+1}|s_{i-1,t},s_{i+1,t})$ are given by

$$P(I|I,I) = 1, \tag{98}$$

$$P(A|A,A) = p_2, \tag{99}$$

$$P(A|I,A) = P(A|A,I) = p_1, \tag{100}$$

and $P(I|s_{i-1},s_{i+1})+P(A|s_{i-1},s_{i+1})=1$, where $0\leq p_1\leq 1$ and $0\leq p_2\leq 1$ are two parameters. Equation (98) ensures that the configuration $\ldots,I,I,I,\ldots$ is the absorbing state. The process in Eq. (99) describes the creation of inactive spots within active islands with probability $1-p_2$. The random walk of boundaries between active and inactive domains is realized by the processes in Eq. (100). A DP transition can be observed only if $p_1>\frac{1}{2}$, when active islands are biased to grow (Wolfram, 1983). The phase diagram of the one-dimensional Domany-Kinzel model is shown in Fig. 6. It comprises an active and an inactive phase, separated by a phase transition line (solid line) belonging to the DP class. The dashed lines corresponds to *directed-bond-percolation* $[p_2=p(2-p_1)]$ and *directed-site-percolation* $(p_1=p_2)$ models. At the special symmetry end point $(p_1=\frac{1}{2},p_2=1)$ compact domain growth (compact directed percolation) occurs and the transition becomes first order (see Sec. IV.C). The transition on the $p_2=0$ axis corresponds to the transition of the stochastic version of Wolfram's rule-18 cellular automaton (Wolfram, 1983). This range-1 SCA generates A at time t only when the right or left neighbor is A at $t-1$:

$$\begin{array}{lll} t-1: & \mathrm{AII} & \mathrm{IIA}, \\ t: & \mathrm{A} & \mathrm{A}, \end{array}$$

with probability p_1 (Boccara and Roger, 1993). The critical point was determined by precise simulations [$p_1^*=0.809\,48(1)$; Vesztergombï *et al.*, 1997]. In the $t\to\infty$ limit the steady state is built up from II and IA blocks (Eloranta and Nummelin, 1992). This finding permits us to map this model onto an even simpler one, the rule-6/16 SCA with new variables: IA→A and II→I:

$$\begin{array}{lllll} t-1: & \mathrm{II} & \mathrm{IA} & \mathrm{AI} & \mathrm{AA}, \\ t: & \mathrm{I} & \mathrm{A} & \mathrm{A} & \mathrm{I}. \end{array}$$

By solving the generalized mean-field approximations and applying Padé approximations (Szabó and Ódor, 1994) or the CAM method (Ódor, 1995), very precise order-parameter exponent estimates were found: $\beta=0.2796(2)$. The damage-spreading phase structure of the one-dimensional Domany-Kinzel model was explored by Hinrichsen *et al.* (1997) and DP-class transitions were found.

Another SCA example is the family of range-4 SCA with an acceptance rule

$$s(t+1,j)=\begin{cases} X & \text{if } y\leq\sum_{j-4}^{j+4}s(t,j)\leq 6 \\ 0 & \text{otherwise}, \end{cases}$$

where $X\in\{0,1\}$ is a two-valued random variable such that $\mathrm{Prob}(X=1)=p$ (Ódor and Szolnoki, 1996). The $y=3$ case was introduced and investigated by Bidaux *et al.* (1989) in $d=1,2,3$. The very first simulations in one dimension (Bidaux *et al.*, 1989) suggested a counterexample to the DP conjecture. More precise spreading simulations of this model (Jensen, 1991), generalized mean-field+coherent-anomaly calculations and simulations of the $y<6$ family in one and two dimensions, have proven that this does not happen for any case (Ódor and Szolnoki, 1996). The transitions either belong to the DP class or are first order.

3. Branching and annihilating random walks with odd number of offspring

Branching and annihilating random walks (BARW's), introduced by Takayasu and Tretyakov (1992), can be regarded as generalizations of the DP process. They are defined by the following reaction-diffusion processes:

$$A\xrightarrow{\sigma}(m+1)A, \quad kA\xrightarrow{\lambda}\varnothing, \quad A\varnothing\overset{D}{\leftrightarrow}\varnothing A. \tag{101}$$

The $2A\to A$ and $2A\to\varnothing$ reactions dominating in the inactive phase have been shown to be equivalent (Peliti, 1986; see Sec. IV.C.1). Therefore the $k=2$ and $m=1$ (BARW1) model differs from the DP process (81) in that spontaneous annihilation of particles is not allowed. The action of the BARW process was established by Cardy and Täuber (1996, 1998) as

$$S=\int d^dxdt[\psi(\partial_t-D\nabla^2)\phi-\lambda(1-\psi^k)\phi^k + \sigma(1-\psi^m)\psi\phi]. \tag{102}$$

A bosonic RG analysis of BARW systems (Cardy and Täuber, 1996) proved that for $k=2$ all the lower branching reactions with $m-2,m-4,\ldots$ are generated via fluctuations involving combinations of branching and annihilation processes. As a consequence, for odd m the $A\to\varnothing$ reaction appears (via $A\to 2A\to\varnothing$). Therefore after the first coarse-graining step in the BARW1 [and in general in odd-m BARW (BARWo) cases], the action becomes the same as that of the DP process. This prediction was confirmed by simulations (see, for example, Jensen, 1993b).

For even m (BARWe), when the parity of the number of particles is conserved, the spontaneous decay $A\to\emptyset$ is not generated. Hence there is an absorbing state with a lone wandering particle. This systems exhibits a non-DP-class critical transition, which will be discussed in Sec. IV.D.1.

4. Directed percolation with spatial boundary conditions

For a review of critical behavior at the surfaces of *equilibrium* models, see Iglói *et al.* (1993). Cardy (1983a) suggested that surface critical phenomena may be described by introducing an additional surface exponent for the order-parameter field, which is generally independent of the other bulk exponents. In nonequilibrium statistical physics one can introduce spatial, temporal (see Sec. IV.A.7), or mixed (see Sec. IV.A.5) boundary conditions.

In directed percolation, an absorbing wall may be introduced by cutting all bonds (the inactive boundary condition) crossing a given $(d-1)$-dimensional hyperplane in space. In the case of reflecting boundary conditions, the wall acts like a mirror, so that the sites within the wall are always a mirror image of those next to the wall. A third type of boundary condition is the active boundary condition, in which sites within the wall are forced to be active. The density at the wall is found to scale as

$$\rho_s^s \sim (p-p_c)^{\beta_1} \tag{103}$$

with a surface critical exponent $\beta_1>\beta$. Owing to the time-reversal symmetry of directed percolation [see Eq. (88)], only one extra exponent is needed to describe surface effects, hence the cluster survival exponent is the same,

$$\beta_1' = \beta_1. \tag{104}$$

The mean lifetime of finite clusters at the wall is defined as

$$\langle t\rangle \sim |\Delta_s|^{-\tau_1}, \tag{105}$$

where $\Delta_s=(p-p_c)$, and is related to β_1 by the scaling relation

$$\tau_1 = \nu_\parallel - \beta_1. \tag{106}$$

The average size of finite clusters grown from seeds on the wall is

$$\langle s\rangle \sim |\Delta_s|^{-\gamma_1}. \tag{107}$$

Series expansions (Essam *et al.*, 1996) and numerical simulations (Lauritsen *et al.*, 1997) in $1+1$ dimensions indicate that the presence of the wall alters several exponents. However, the scaling properties of the correlation lengths (as given by $\nu_\parallel$ and $\nu_\perp$) are *not* altered.

The field theory for directed percolation in a semi-infinite geometry was first analyzed by Janssen *et al.* (1988). They showed that the appropriate action for directed percolation with a wall at $x_\perp=0$ is given by $S=S_{\text{bulk}}+S_{\text{surface}}$, where

$$S_{\text{surface}} = \int d^{d-1}x \int dt \Delta_s \psi_s \phi_s, \tag{108}$$

with the definitions $\phi_s=\phi(\mathbf{x}_\parallel, x_\perp=0,t)$ and $\psi_s=\psi(\mathbf{x}_\parallel, x_\perp=0,t)$. The surface term S_{surface} corresponds to the most relevant interaction consistent with the symmetries of the problem and the one that also respects the absorbing-state condition. The appropriate surface exponents were computed to first order in $\epsilon=4-d$ using renormalization-group techniques:

$$\beta_1 = \frac{3}{2} - \frac{7\epsilon}{48} + O(\epsilon^2). \tag{109}$$

These calculations also showed that the corresponding hyperscaling relation is

$$\nu_\parallel + d\nu_\perp = \beta_1 + \beta + \gamma_1, \tag{110}$$

relating β_1 to

$$\gamma_1 = \frac{1}{2} + \frac{7\epsilon}{48} + O(\epsilon^2). \tag{111}$$

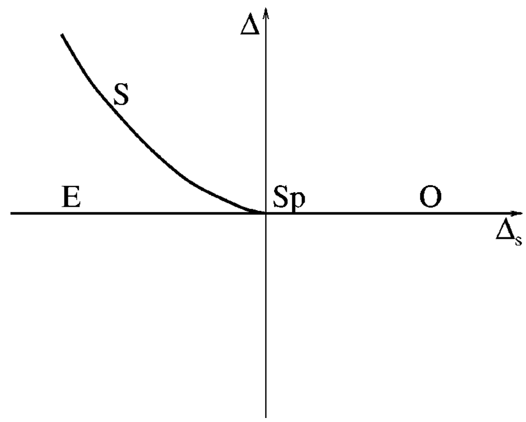

FIG. 7. Schematic mean-field phase diagram for boundary directed percolation. The transitions are labeled by O =ordinary, E=extraordinary, S=surface, and Sp=special. From Fröjdh *et al.*, 2001.

The schematic phase diagram for directed percolation at the boundary is shown in Fig. 7, where Δ and Δ_s represent the deviations of the bulk and surface, respectively, from criticality. For $\Delta>0$ and for Δ_s sufficiently negative the boundary is ordered even while the bulk is disordered, and there is a *surface transition*. For $\Delta_s<0$ and $\Delta\to 0$, the bulk is ordered in the presence of an already ordered boundary, and there is an *extraordinary transition* of the boundary. Finally at $\Delta=\Delta_s=0$, where all the critical lines meet, and where both the bulk and isolated surface are critical, we find a multicritical point, i.e., a *special transition*.

For $\Delta_s>0$ and $\Delta\to 0$ there is an ordinary transition, since the bulk orders in a situation in which the boundary, if isolated, would be disordered. At the ordinary

TABLE XIII. Critical exponents for directed percolation in $d=1$ and $d=2$ for the ordinary transition at the boundary.

	$d=1$	$d=2$	Mean field
β_1	0.733 71(2)	1.07(5)	3/2
$\delta_1=\alpha_1$	0.423 17(2)	0.82(4)	3/2
τ_1	1.000 14(2)	0.26(2)	0
γ_1	1.820 51(1)	1.05(2)	1/2

transition, one finds just one extra independent exponent related to the boundary. This can be the surface density exponent $\beta_{1,\mathrm{dens}}$. In one dimension the two cases of inactive and reflecting boundaries belong to the same universality class (Lauritsen *et al.*, 1998) as was identified as the ordinary transition. There are numerical data for the exponents of the extraordinary and special transitions (however, see Janssen *et al.*, 1988 for an RG analysis).

The best exponent estimates currently available were summarized by Fröjdh *et al.* (2001). Some of them are shown in Table XIII. In $d=1$ the best results are from series expansions (Essam *et al.*, 1996; Jensen, 1999b); in all other cases the best results are from Monte Carlo data (Lauritsen *et al.*, 1997, 1998; Fröjdh *et al.*, 1998; Howard *et al.*, 2000). The exponent τ_1 was conjectured to equal unity (Essam *et al.*, 1996), although this has now been challenged by the estimate $\tau_1=1.000\,14(2)$ (Jensen, 1999b).

It has been known for some time that the presence of an edge introduces new exponents, independent of those associated with the bulk or with a surface (see Cardy, 1983a). For an investigation showing numerical estimates in two-dimensional and mean-field values, see Fröjdh *et al.* (1998). Table XIV summarizes results for the ordinary edge exponents. A closely related application is the study of spreading processes in narrow channels (Albano, 1997).

5. Directed percolation with mixed (parabolic) boundary condition scaling

Boundary conditions, which act in both space and time have also been investigated in dynamical systems. These turn out to be related to the hard-core repulsion effects of one-dimensional systems (Sec. V.B). Kaiser and Turban (1994, 1995) investigated the (1+1)-dimensional DP process confined in a parabola-shaped geometry. Assuming an absorbing boundary of the form

TABLE XIV. Numerical estimates for the ordinary β_2^{O} exponents for edge directed percolation together with the mean-field values. Note that $\beta_2^{\mathrm{O}}(\pi)=\beta_1^{\mathrm{O}}$.

Angle (α)	$\pi/2$	$3\pi/4$	π	$5\pi/4$
β_2^{O} ($d=2$)	1.6(1)	1.23(7)	1.07(5)	0.98(5)
β_2^{O} (MF)	2	5/3	3/2	7/5

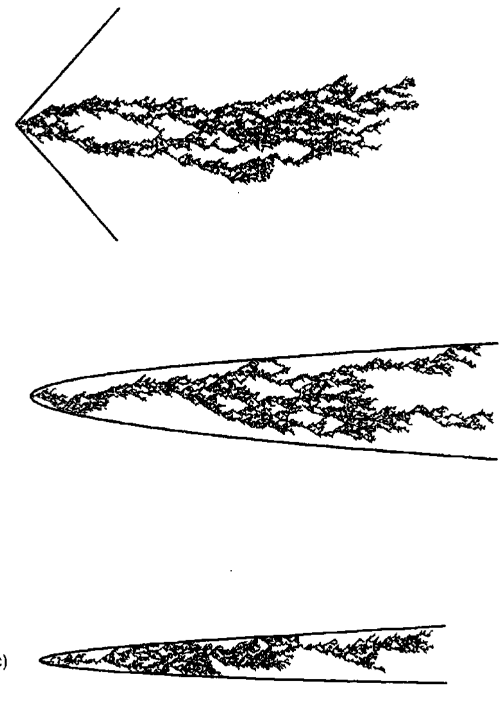

FIG. 8. The space-time evolution of the critical, (1+1)-dimensional directed site percolation process confined by a parabola (a) $\sigma<1/Z$, (b) $\sigma=1/Z$, (c) $\sigma>1/Z$. From Kaiser and Turban, 1994.

$x=\pm Ct^{\sigma}$ they proposed a general scaling theory. It is based on the observation that the coefficient of the parabola (C) scales as $C\to\Lambda^{Z\sigma-1}C$ under rescaling,

$$x\to\Lambda x,\quad t\to\Lambda^{Z}t,\quad \Delta\to\Lambda^{-1/\nu_\perp}\Delta,\quad \rho\to\Lambda^{-\beta/\nu_\perp}\rho, \tag{112}$$

where $\Delta=|p-p_c|$ and Z is the dynamical exponent of DP. By a conformal mapping of the parabola to straight lines and using the mean-field approximation, they claimed that this boundary is a relevant perturbation for $\sigma>1/Z$, irrelevant for $\sigma<1/Z$, and marginal for $\sigma=1/Z$ (see Fig. 8). The marginal case results in C-dependent nonuniversal power-law decay, while for the relevant case stretched exponential functions have been obtained. Kaiser and Turban have given support to these claims by numerical simulations.

6. Lévy-flight anomalous diffusion in directed percolation

Lévy-flight anomalous diffusion generating long-range correlations has already been mentioned in Sec. III.A.5 and III.B.3 in the case of equilibrium models. In nonequilibrium systems, Grassberger (1986), following the suggestion of Mollison (1977), introduced a variation of the epidemic process with infection probability distribution $P(R)$, which decays with the distance R as a power-law,

$$P(R) \propto \frac{1}{R^{d+\sigma}}. \tag{113}$$

This formalism can model long-range epidemics mediated by flies, wind, etc. Grassberger claimed that the critical exponents should depend continuously on σ. This result was confirmed in one dimension by Marques and Ferreira (1994), who estimated β based on coherent anomaly calculations. The study of anomalous diffusion was extended for generalized epidemic processes (see Sec. IV.B) and for annihilating random walks, too (see Sec. IV.C.4). The effective action of the Lévy-flight DP model is

$$S[\psi,\phi] = \int d^d x dt \Big(\psi(\partial_t - \tau - D_N \nabla^2 - D_A \nabla^\sigma)\phi + \frac{g}{2}(\psi\phi^2 - \psi^2\phi) \Big), \tag{114}$$

where D_N denotes the normal diffusion constant, D_A the anomalous diffusion constant, and g the interaction coupling constant. The field-theoretical RG method up to first order in an $\epsilon = 2\sigma - d$ expansion (Janssen *et al.*, 1999) gives

$$\beta = 1 - \frac{2\epsilon}{7\sigma} + O(\epsilon^2),$$

$$\nu_\perp = \frac{1}{\sigma} + \frac{2\epsilon}{7\sigma^2} + O(\epsilon^2),$$

$$\nu_\| = 1 + \frac{\epsilon}{7\sigma} + O(\epsilon^2),$$

$$Z = \frac{\nu_\|}{\nu_\perp} = \sigma - \epsilon/7 + O(\epsilon^2). \tag{115}$$

Moreover, Marques and Ferreira showed that the hyperscaling relation

$$\eta + 2\delta = d/Z \quad (\delta = \beta/\nu_\|), \tag{116}$$

for the so-called *critical initial slip exponent* η and the relation

$$\nu_\| - \nu_\perp(\sigma - d) - 2\beta = 0, \tag{117}$$

hold exactly for arbitrary values of σ. Numerical simulations on (1+1)-dimensional bond percolation confirmed these results except in the neighborhood of $\sigma = 2$ (Hinrichsen and Howard, 1999; see Fig. 9).

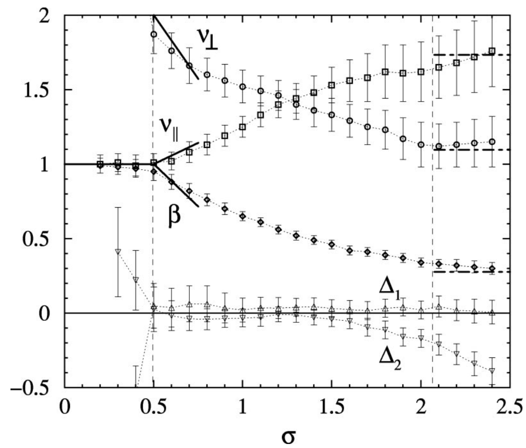

FIG. 9. Estimates for the exponent β and the derived exponents $\nu_\perp$ and $\nu_\|$ in comparison with the field-theoretic results (solid lines) and the DP exponents (dotted lines). The quantities Δ_1 and Δ_2 represent deviations from the scaling relations (116) and (117), respectively. The vertical, dashed lines distinguish between regions of constant and continuously changing exponents. From Hinrichsen and Howard, 1999.

7. Long-range correlated initial conditions in directed percolation

It is well known that initial conditions influence the temporal evolution of nonequilibrium systems. The "memory" of systems for the initial state usually depends on the dynamical rules. For example, stochastic processes with a finite temporal correlation length relax to their stationary state in an exponentially short time. An interesting situation emerges when a system undergoes a nonequilibrium phase transition in which the temporal correlation length diverges. This setup raises the question whether it is possible to construct initial states that affect the *entire* temporal evolution of such systems.

Monte Carlo simulations of critical models with absorbing states usually employ two different types of initial conditions. On the one hand, *uncorrelated random initial conditions* (Poisson distributions) are used to study the relaxation of an initial state with a finite particle density towards an absorbing state. In this case the particle density $\rho(t)$ decreases on the infinite lattice asymptotically as

$$\rho(t) \sim t^{-\beta/\nu_\|}. \tag{118}$$

On the other hand, in spreading simulations (Grassberger and de la Torre, 1979), each run starts with a single particle as a localized active seed from which a cluster originates (this is a long-range correlated state). Although many of these clusters survive for only a short time, the number of particles $n(t)$ averaged over many independent runs increases as

$$\langle n(t) \rangle \sim t^{+\eta}. \tag{119}$$

These two cases seem to represent extremal situations in which the average particle number either decreases or increases.

A crossover between these two extremal cases takes place in a critical spreading process that starts from a random initial condition of very low density. Here the particles are initially separated by empty intervals of a certain typical size, so that the average particle number first increases according to Eq. (119). Later, when the growing clusters begin to interact with each other, the system crosses over to the algebraic decay of Eq. (118)—a phenomenon referred to as the "critical initial slip" of a nonequilibrium system (Janssen *et al.*, 1989).

Hinrichsen and Ódor (1998) and Menyhárd and Ódor (2000) investigated whether it is possible to interpolate continuously between the two extremal cases in (1+1)-dimensional DP and parity-conserving processes. It was shown that one can in fact generate certain initial states in such a way that the particle density on the infinite lattice varies as

$$\rho(t) \sim t^{\kappa} \tag{120}$$

with a continuously adjustable exponent κ in the range

$$-\beta/\nu_{\|} \leqslant \kappa \leqslant +\eta. \tag{121}$$

To this end artificial initial configurations with algebraic long-range correlations of the form

$$C(r) = \langle s_i s_{i+r} \rangle \sim r^{-(d-\sigma)} \tag{122}$$

were constructed, where $\langle\,\rangle$ denotes the average over many independent realizations, d the spatial dimension, and $s_i = 0,1$ inactive and active sites. The exponent σ is a free parameter and can be varied continuously between 0 and 1. This initial condition can be taken into account by adding the term

$$S_{ic} = \mu \int d^d x \psi(x,0) \phi_0(x) \tag{123}$$

to the action, where $\phi_0(x)$ represents the initial particle distribution. The long-range correlations limit $\sigma \to d$ corresponds to a constant particle density, and thus we expect Eq. (118) to hold [$\phi_0(x) = \text{const}$ is irrelevant under rescaling]. On the other hand, the short-range limit $\sigma \to 0$ represents an initial state in which active sites are separated by infinitely large intervals [$\phi_0(x) = \delta^d(x)$] so that the particle density should increase according to Eq. (119). In between we expect $\rho(t)$ to vary algebraically according to Eq. (120) with an exponent κ depending continuously on σ.

In case of the (1+1)-dimensional Domany-Kinzel SCA (see Sec. IV.A.2), field-theoretical renormalization-group calculation and simulations have proved (Hinrichsen and Ódor, 1998) the exact functional dependence,

$$\kappa(\sigma) = \begin{cases} \eta & \text{for } \sigma < \sigma_c \\ \frac{1}{z}(d - \sigma - \beta/\nu_\perp) & \text{for } \sigma > \sigma_c, \end{cases} \tag{124}$$

with the critical threshold $\sigma_c = \beta/\nu_\perp$.

8. Quench-disordered directed percolation systems

Perhaps the lack of experimental observation of a robust DP class is owing to the fact that even weak disorder changes the critical behavior of such models. Therefore this section provides a view onto interesting generalizations of DP processes that may be observable in physical systems. First, Noest (1986, 1988) showed using the Harris criterion (Harris, 1974a) that spatially quenched disorder (frozen in space) changes the critical behavior of DP systems for $d < 4$. Then Janssen (1997a) studied the problem by field theory, taking into account the disorder in the action by adding the term

$$S \to S + \gamma \int d^d x \left[\int dt \psi \phi \right]^2. \tag{125}$$

This additional term causes marginal perturbation, and the stable fixed point is shifted to an unphysical region, leading to runaway solutions of the flow equations in the physical region of interest, meaning that spatially quenched disorder changes the critical behavior of directed percolation. This conclusion is supported by the simulation results of Moreira and Dickman (1996), who reported logarithmic spreading behavior in the two-dimensional contact process at criticality. In the subcritical region they found Griffiths phase, in which the time dependence is governed by nonuniversal power laws, while in the active phase the relaxation of $P(t)$ is algebraic.

In 1+1 dimension, Noest (1986, 1988) predicted generic scale invariance. Webman *et al.* (1998) reported a glassy phase with nonuniversal exponents in a (1+1)-dimensional DP process with quenched disorder. Cafiero *et al.* (1998) showed that directed percolation with spatially quenched randomness in the large-time limit can be mapped onto a non-Markovian spreading process with memory, in agreement with previous results. They also showed that the time-reversal symmetry of the DP process [Eq. (88)] is not broken. Therefore

$$\delta = \delta', \tag{126}$$

and they were able to derive a hyperscaling law for the inactive phase,

$$\eta = dz/2, \tag{127}$$

and for the absorbing phase,

$$\eta + \delta = dz/2. \tag{128}$$

Webman *et al.* confirmed these relations by simulations and found that the dynamical exponents changed continuously as a function of the disorder probability. An RG study by Hooyberghs *et al.* (2003) showed that in the case of strong enough disorder the critical behavior is controlled by an infinite-randomness fixed point, the static exponents of which in one dimension are

$$\beta = (3 - \sqrt{5})/2, \quad \nu_\perp = 2, \tag{129}$$

and $\xi^{1/2} \propto \ln \tau$. For disorder strengths outside the attractive region of the fixed point disorder strength dependent critical exponents were found.

Temporally quenched disorder can be taken into the action by adding the term

$$S \to S + \gamma \int dt \left[\int d^d x \psi \phi \right]^2. \tag{130}$$

This is a relevant perturbation for the DP processes. Jensen (1996b) investigated the (1+1)-dimensional directed bond percolation (see Sec. IV.A.2) with temporal disorder via series expansions and Monte Carlo simulations. The temporal disorder was introduced by allowing time slices to become fully deterministic ($p_1 = p_2 = 1$), with probability α. Jensen found α-dependent, continuously changing critical point and critical exponent values between those of the (1+1)-dimensional DP class and those of the deterministic percolation. This latter class is defined by the exponents

$$\beta=0,\ \delta=0,\ \eta=1,\ Z=1,\ \nu_{\|}=2,\ \nu_{\perp}=2. \tag{131}$$

For small values of the disorder parameter, violation of the Harris criterion was reported.

If quenched disorder takes place in both space and time, the corresponding term added to the action is

$$S \to S + \gamma \int dtd^dx[\psi\phi]^2, \tag{132}$$

which is an irrelevant perturbation to the Reggeon field theory. This has the same properties as the intrinsic noise in the system and can be considered as being annealed.

B. Dynamical percolation classes

If we allow memory in the unary DP spreading process (Sec. IV.A) such that the infected sites may have a different reinfection probability (p) from the virgin ones (q), we obtain different percolation behavior (Grassberger *et al.*, 1997). The model in which the reinfection probability is zero is called the *general epidemic process* (Mollison, 1977). In this case the epidemic stops in finite systems but an infinite epidemic is possible in the form of a single wave of activity. When starting from a single seed this leads to annular growth patterns. The transition between survival and extinction is a critical phenomenon called dynamical percolation (DYP; Grassberger, 1982b). Clusters generated at criticality are the ordinary percolation clusters of the lattice in question. Field-theoretical treatments have been given by Cardy (1983b); Cardy and Grassberger (1985); Janssen (1985); Muñoz *et al.* (1998); Jimenez-Dalmaroni and Hinrichsen (2003). The action of the model is

$$S = \int d^dx dt \frac{D}{2}\psi^2\phi - \psi\left(\partial_t\phi - \nabla^2\phi - r\phi + w\phi\int_0^t ds\phi(s)\right). \tag{133}$$

This action is invariant under the nonlocal symmetry transformation

$$\phi(x,t) \leftrightarrow -\partial_t\psi(x,-t), \tag{134}$$

which results in the hyperscaling relation (Muñoz *et al.*, 1997):

$$\eta + 2\delta + 1 = \frac{dz}{2}. \tag{135}$$

As in case of directed percolation, the relations $\beta=\beta'$ and $\delta=\alpha$ again hold. The upper critical dimension is $d_c=6$. The dynamical critical exponents, as well as spreading and avalanche exponents, are summarized by Muñoz *et al.* (1999). The dynamical exponents are $Z=1.1295$ for $d=2$, $Z=1.336$ for $d=3$, and $Z=2$ for $d=6$. Dynamical percolation has been observed in forest fire models (Drossel and Schwabl, 1993; Albano, 1994) and in some Lotka-Volterra-type lattice prey-predator models (Antal *et al.*, 2001).

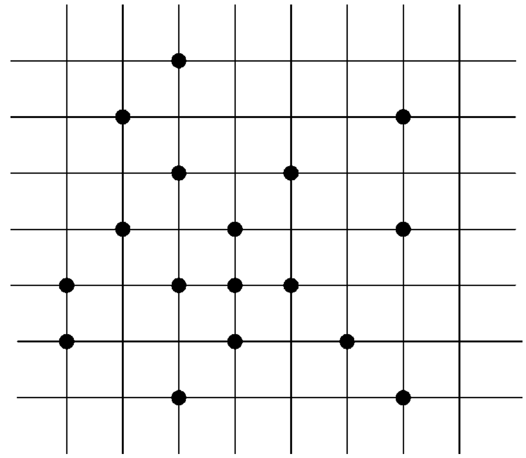

FIG. 10. Isotropic site percolation in $d=2$ dimensions.

1. Isotropic percolation universality classes

Ordinary percolation (Stauffer and Aharony, 1994; Grimmett, 1999) is a geometrical phenomenon that describes the occurrence of infinitely large connected clusters by *completely random* displacement of some variables (sites, bonds, etc., with probability p) on lattices (see Fig. 10).

The dynamical percolation process is known to generate such percolating clusters (see Sec. IV.B). At the transition point, moments of the s cluster size distribution $n_s(p)$ show singular behavior. Ordinary percolation corresponds to the $q=1$ limit of the Potts model. That means its generating functions can be expressed in terms of the free energy of the $q\to 1$ Potts model. In the low-temperature dilute Ising model, the occupation probability (p) driven magnetization transition is an ordinary percolation transition, as well. As a consequence the critical exponents of the magnetization can be related to the cluster-size exponents. For example, the susceptibility obeys simple homogeneity form with $p-p_c$ replacing $T-T_c$,

$$\chi \propto |p-p_c|^{-\gamma}. \tag{136}$$

Table XV summarizes the known critical exponents of ordinary percolation. The exponents are from the overview of Bunde and Havlin (1991). A field-theoretical treatment by Benzoni and Cardy (1984) provided an upper critical dimension $d_c=6$. The $d=1$ case is special: $p_c=1$ and the order parameter jumps ($\beta=0$). Furthermore, here some exponents exhibit nonuniversal behavior by increasing the interaction length unless we redefine the scaling variable (see Stauffer and Aharony, 1994).

TABLE XV. Critical exponents of ordinary percolation.

d	$\beta=\beta'$	γ_p	$\nu_\perp$	σ	τ
1	0	1	1	1	2
2	5/36	43/18	4/3	36/91	187/91
3	0.418(1)	1.793(4)	0.8765(17)	0.452(1)	2.189(1)
4	0.64	1.44	0.68	0.48	2.31
5	0.84	1.18	0.57	0.49	2.41
6	1	1	1/2	1/2	3/2

2. Dynamical percolation with spatial boundary conditions

There are very few numerical results for surface critical exponents of dynamical percolation. The generalized epidemic process in three dimensions was investigated numerically by Grassberger (1992b). The surface and edge exponents (for angle $\pi/2$) were determined in the case of inactive boundary conditions. Different measurements (density and cluster simulations) resulted in a single surface $\beta_1=0.848(6)$ and a single edge $\beta_e=1.36(1)$ exponent.

3. Lévy-flight anomalous diffusion in dynamic percolation

To model long-range epidemic spreading in a system with immunization, the effect of Lévy-flight diffusion [Eq. (113)] was investigated by Janssen *et al.* (1999). The renormalization-group analysis of the generalized epidemic process with anomalous diffusion resulted in the following four $\epsilon=3\sigma-d$ expansion results: (a) for the critical initial slip exponent,

$$\eta=\frac{3\epsilon}{16\sigma}+O(\epsilon^2); \tag{137}$$

(b) for the order-parameter [density of removed (immune) individuals] exponent,

$$\beta=1-\frac{\epsilon}{4\sigma}+O(\epsilon^2); \tag{138}$$

(c) for the spatial correlation exponent,

$$\nu_\perp=\frac{1}{\sigma}+\frac{\epsilon}{4\sigma^2}+O(\bar{\varepsilon}^2); \tag{139}$$

and (d) for the temporal correlation length exponent,

$$\nu_\parallel=1+\frac{\epsilon}{16\sigma}+O(\epsilon^2). \tag{140}$$

C. Voter model classes

Now we turn to models that can describe the spreading of voter opinion, arranged on regular lattices. These models exhibit first-order transitions but dynamical scaling can still be observed in them. The voter model (Ligget, 1985; Durret, 1988) is defined by the following spin-flip dynamics. A site is selected randomly which takes the "opinion" (or spin) of one of its nearest neighbors (with probability p). This rule ensures that the model has two homogeneous absorbing states (all spin up or down) and is invariant under the Z_2 symmetry. A general feature of these models is that dynamics takes place only at the boundaries. The action that describes this behavior, proposed by Dickman and Tretyakov (1995) and Peliti (1986),

$$S=\int d^dxdt\left[\frac{D}{2}\phi(1-\phi)\psi^2-\psi(\partial_t\phi-\lambda\nabla^2\phi)\right], \tag{141}$$

is invariant under the symmetry transformation

$$\phi\leftrightarrow 1-\phi,\quad \psi\leftrightarrow-\psi. \tag{142}$$

This results in the "hyperscaling" relation (Muñoz *et al.*, 1997)

$$\delta+\eta=dz/2, \tag{143}$$

which is valid for all first-order transitions ($\beta=0$) with $d\leqslant 2$, hence $d_c=2$ is the upper critical dimension. It is also valid for all *compact growth processes* (where "compact" means that the density in surviving colonies remains finite as $t\to\infty$).

In one dimension at the upper terminal point of the Domany-Kinzel SCA (Fig. 6, $p_1=\frac{1}{2}$, $p_2=1$), an extra Z_2 symmetry exists between 1's and 0's, hence the scaling behavior is not DP-class-like but corresponds to the fixed point of the inactive phase of parity-conserving class models (Sec. IV.D). As a consequence compact domains of 0's and 1's grow such that the domain walls follow annihilating random walks (ARW) (see Sec. IV.C.1) and belong to the one-dimensional voter model class. In one dimension compact directed percolation is also equivalent to the $T=0$ Glauber Ising model (see Secs. II and III.A). When nonzero temperature is applied (corresponding to spin flips in domains) or symmetry is broken (by changing p_2 or adding an external magnetic field), a first-order transition takes place ($\beta=0$).

In two and higher dimensions the $p=1$ situation corresponds to the $p_1=3/4$, $p_2=1$ point in the phase diagram of Z_2-symmetric models (see Fig. 1). This model has a "duality" symmetry with coalescing random walks: going backward in time, the successive ancestors of a given spin follow the trail of a simple random walk; comparing the values of several spins shows that their associated random walks necessarily merge when they meet (Ligget, 1985). This correspondence permits us to solve many aspects of the kinetics. In particular, the calculation of the density of interfaces $\rho_m(t)$ (i.e., the fraction of $+-$ nearest-neighbor pairs) starting from random initial conditions of magnetization m, is ultimately given by the probability that a random walk initially at unit distance from the origin has not yet reached it at time t. Therefore, owing to the recurrence properties of random walks, the voter model shows coarsening for $d\leqslant 2$ [i.e., $\rho_m(t)\to 0$ when $t\to\infty$]. For the "marginal" $d=2$ case one

TABLE XVI. Critical exponents of voter model classes.

d	β	β'	γ	$\nu_\parallel$	$\nu_\perp$	Z	δ	η
1	0.0	1	2	2	1	2	1/2	0
2	0.0	1	1	1	1/2	2	1	0

finds a slow logarithmic decay (Scheucher and Spohn, 1988; Krapivsky, 1992; Frachebourg and Krapivsky, 1996):

$$\rho_m(t) = (1-m^2)\left[\frac{2\pi D}{\ln t} + \mathcal{O}\left(\frac{1}{\ln^2 t}\right)\right], \tag{144}$$

with D being the diffusion constant of the underlying random walk ($D=1/4$ for the standard case of nearest-neighbor, square lattice walks, when each spin is updated on average once per unit of time).

Simulating general, Z_2-symmetric spin-flip rules in two dimensions, Dornic *et al.* (2001) conjectured that all critical Z_2-symmetric rules without bulk noise form a codimension-1 "voterlike" manifold separating order from disorder, characterized by the logarithmic decay of both ρ and m. The critical exponents for this class are summarized in Table XVI. Furthermore Dornic *et al.* (2001) found that this Z_2 symmetry is not a necessary condition; voter model behavior can also be observed in systems without bulk fluctuations, where the total magnetization is conserved. A field-theoretical understanding of these results is still lacking.

1. The $2A\to\varnothing$ and the $2A\to A$ models

As mentioned in Sec. IV.C, in one dimension the annihilating random walk (ARW) and voter models are equivalent. In higher dimensions this is not the case (see Sec. IV.E). The simplest reaction-diffusion model—in which identical particles follow a random walk and annihilate on contact of a pair—is adequately described by mean-field-type equations in $d_c>2$ dimensions,

$$\rho(t) \propto t^{-1}, \tag{145}$$

but in lower dimensions fluctuations become relevant. Omitting boundary and initial condition terms, the field-theoretical action is

$$S = \int d^dx dt[\psi(\partial_t\phi - D\nabla^2\phi) - \lambda(1-\psi^2)\phi^2], \tag{146}$$

where D denotes the diffusion coefficient and λ is the annihilation rate.

For $d=d_c=2$ the leading-order decay of the ARW was derived exactly by Lee using the field-theoretical RG method (Lee, 1994):

$$\rho(t) = \frac{1}{8\pi D}\ln(t)/t + O(1/t). \tag{147}$$

For $d=1$, Rácz (1985) and Lushnikov (1987) predicted that the particle density decays as

$$\rho(t) = A_2(Dt)^{-1/2}. \tag{148}$$

This scaling law was confirmed by ϵ expansion and the universal amplitude A_2 was found to be

$$\frac{1}{4\pi\epsilon} + \frac{2\ln 8\pi - 5}{16\pi} + O(\epsilon). \tag{149}$$

The universal scaling behavior of the ARW was shown to be equivalent to that of the $A+A\to A$ coagulation random-walk process by Peliti (1986). The renormalization-group approach provided universal decay amplitudes (different from those of the ARW) to all orders in the epsilon expansion. It was also shown (Domany and Kinzel, 1984) that the motion of kinks in the compact version of directed percolation (Essam, 1989) and the Glauber-Ising model (Glauber, 1963) at the $T=0$ transition point are also described exactly by Eq. (148). These reactions also have an intimate relationship to the Edwards-Wilkinson interface growth model (see Sec. VI.B).

2. Compact directed percolation with spatial boundary conditions

By introducing a wall in compact directed percolation, one alters the survival probability and obtains surface critical exponents just as for DP. With inactive boundary conditions, the cluster is free to approach and leave the wall, but not to cross it. For $d=1$, this gives rise to $\beta_1'=2$. On the other hand, for active boundary conditions, the cluster is stuck to the wall and therefore described by a single random walker for $d=1$. By reflection in the wall, this may be viewed as *symmetric compact directed percolation*, which has the same β' as normal compact DP, giving $\beta_1'=1$ (Essam and TanlaKishani, 1994; Essam and Guttmann, 1995).

3. Compact directed percolation with parabolic boundary conditions

Space-time boundaries are also of interest in compact directed percolation. Cluster simulations in $1+1$ dimension and mean-field approximations (Ódor and Menyhárd, 2000; Dickman and ben Avraham, 2001) for percolation confined by repulsive parabolic boundary condition of the form $x=\pm Ct^\sigma$ resulted in C-dependent δ and η exponents (see Fig. 11) similarly to the DP case (see Sec. IV.A.5) with the marginal condition: $\sigma=1/2$. In the mean-field approximation (Ódor and Menyhárd, 2000) results similar to those for directed percolation were obtained (Kaiser and Turban, 1995). Analytical results can be obtained only in limiting cases. For narrow systems (small C) one obtains the following asymptotic behavior for the connectedness function to the origin:

$$P(t,x) \sim t^{-\pi^2/8C^2}\cos\left(\frac{\pi x}{2C\sqrt{t}}\right). \tag{150}$$

Recently an analytical solution was derived for a related problem (Dickman and ben Avraham, 2001). For a one-dimensional lattice random walk with an absorbing

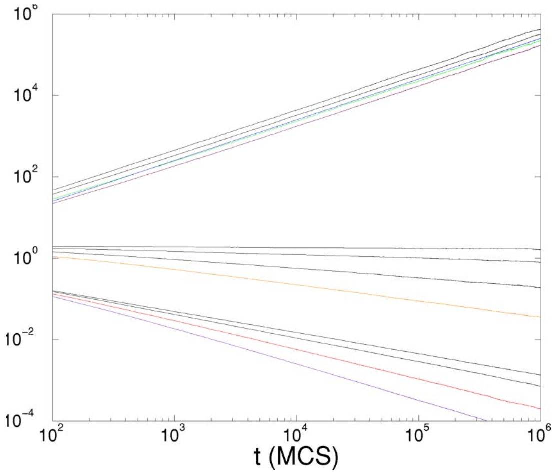

FIG. 11. (Color in online edition) Parabola boundary confinement cluster simulations for compact directed percolation. Middle curves: number of active sites (C=2,1.5,1.2,1, top to bottom); lower curves: survival probability (C=2,1.5,1.2,1, top to bottom); upper curves: $R^2(t)$ (C=2,1.5,1.2,1, top to bottom). From Ódor and Menyhárd, 2000.

boundary at the origin and a movable partial reflector (with probability r) δ varies continuously between 1/2 and 1 as r varies between 0 and 1.

4. Lévy-flight anomalous diffusion in annihilating random walks

Long-range interactions generated by nonlocal diffusion in annihilating random walks result in the recovery of mean-field behavior. This has been studied by different approaches. Particles performing simple random walks subject to the reactions $A+B\to\emptyset$ (Sec. IV.C.1) and $A+A\to\emptyset$ (Sec. IV.A) in the presence of a quenched velocity field were investigated by Zumofen and Klafter (1994). The quenched velocity field enhances the diffusion in such a way that the effective action of the velocity field is reproduced if Lévy flights are substituted for the simple random-walk motion. In the above-mentioned reactions, particle density decay is algebraic with an exponent related to the step length distribution of the Lévy flights defined in Eq. (113). These results have been confirmed by several renormalization-group calculations (Oerding, 1996; Deem and Park, 1998a, 1998b).

The $A+A\to\emptyset$ process with anomalous diffusion was investigated by field theory (Hinrichsen and Howard, 1999). The action of this model is

$$S[\bar{\psi},\psi] = \int d^d x dt \{\bar{\psi}(\partial_t - D_N\nabla^2 - D_A\nabla^{\sigma})\psi + 2\lambda\bar{\psi}\psi^2 + \lambda\bar{\psi}^2\psi^2 - n_0\bar{\psi}\delta(t)\}, \tag{151}$$

where n_0 is the initial (homogeneous) density at t=0. The density decays for $\sigma<2$ as

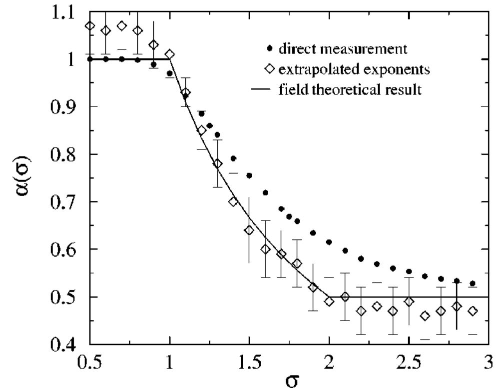

FIG. 12. The anomalous annihilation process: the graph from Hinrichsen and Howard (1999) shows direct estimates and extrapolations for the decay exponent α, as a function of σ. The solid line represents the exact result (neglecting log corrections at σ=1).

$$n(t) \sim \begin{cases} t^{-d/\sigma} & \text{for } d<\sigma, \\ t^{-1}\ln t & \text{for } d=d_c=\sigma, \\ t^{-1} & \text{for } d>\sigma. \end{cases} \tag{152}$$

The simulation results of the corresponding (1+1)-dimensional lattice model (Hinrichsen and Howard, 1999) can be seen in Fig. 12. It was also shown by Hinrichsen and Howard (1999) that Lévy-flight annihilation and coagulation processes ($A+A\to A$) are in the same universality class.

D. Parity-conserving classes

In an attempt to generalize DP and compact-DP-like systems, we turn to new models, in which a conservation law is relevant. A new universality class appears among (1+1)-dimensional, single-component, reaction-diffusion models. Although it is usually named the parity-conserving class, examples have proved that parity conservation alone is not a sufficient condition for the behavior of this class. For example, Inui *et al.* (1995) showed that a one-dimensional stochastic cellular automaton with global parity conservation exhibited a DP-class transition. The binary spreading process (see Sec. V.F) in one (Park *et al.*, 2001) and two dimensions (Ódor *et al.*, 2002) were also found to be insensitive to the presence of parity conservation. Multicomponent BARW2 models in one dimension (see Sec. V.K) generate different, robust classes again (Cardy and Täuber, 1996; Hooyberghs *et al.*, 2001; Ódor, 2001b). Today it is known that the BARW2 dynamics in single-component, single-absorbing-state systems (without inhomogeneities, long-range interactions, or other symmetries) provide sufficient conditions for the parity-conserving class (Cardy and Täuber, 1996). In single-component, multiabsorbing state systems, Z_2 symmetry is a necessary but not sufficient condition—the BARW2 dynamics (Sec. IV.D.1) of domain walls is also a necessary condition. Some studies have shown (Park and Park, 1995; Menyhárd and Ódor, 1996; Hwang *et al.*, 1998) that an external field that de-

TABLE XVII. Critical exponents of branching and annihilating random walks with even offspring (BARWe).

d	β	β'	γ	δ	Z	$\nu_{\|}$	η
1	0.92(3)	0.92(3)	0.00(5)	0.285(2)	1.75	3.25(10)	0.000(1)
2	1	0	1	0	2	1	−1/2

stroys the Z_2 symmetry of absorbing states (but preserves the BARW2 dynamics) yields a DP-class instead of a parity-conserving-class transition in the system. Some other names for this class are also used, like the directed Ising (DI) class, or the BARW class.

1. Branching and annihilating random walks with even number of offspring

Generic models of the parity-conserving class for which field-theoretical treatments exist are the branching and annihilating random-walk models—introduced in Sec. IV.A.3—with $k=2$ and even number (m) of offspring (BARWe; Jensen, 1993c, 1994; ben Avraham *et al.*, 1994; Zhong and Avraham, 1995; Lipowski, 1996; Janssen, 1997b). These conserve the particle number mod 2 and hence have two distinct sectors, an odd- and an even-parity one. In the even sector, particles finally die out ($\delta\neq 0, \eta=0$), while in the odd sector at least one particle always remain alive ($\delta=0, \eta\neq 0$). BARWe dynamics may also appear in multicomponent models exhibiting Z_2-symmetric absorbing states in terms of the kinks between ordered domains. Examples of such systems are the nonequilibrium kinetic Ising model (Sec. IV.D.2) and generalized Domany-Kinzel model (Sec. IV.D.3). It has been conjectured (Menyhárd and Ódor, 2000) that in all models with Z_2-symmetric absorbing states an underlying BARWe process is a necessary condition for phase transitions with parity-conserving criticality. Sometimes it is not so easy to find the underlying BARWe process, which can be seen on the coarse-grained level only (in the generalized Domany-Kinzel model, for instance, the kinks are spatially extended objects). This might have led some studies to the conjecture that Z_2 symmetry is a sufficient condition for the parity-conserving class (Hwang and Park, 1999). However, the example of compact directed percolation (see Sec. IV.C) shows that this cannot be true. The field theory of BARWe models was investigated by Cardy and Täuber (1996, 1998). For this case the action

$$S=\int d^dxdt[\psi(\partial_t - D\nabla^2)\phi - \lambda(1-\psi^2)\phi^2 + \sigma(1-\psi^2)\psi\phi] \quad (153)$$

is invariant under the simultaneous transformation of fields,

$$\psi \leftrightarrow -\psi, \quad \phi \leftrightarrow -\phi. \quad (154)$$

Owing to the nonrecurrence of random walks in $d\geqslant 2$, the system is in the active phase for $\sigma>0$ and a mean-field transition occurs with $\beta=1$. However, the survival probability of a particle cluster is finite for any $\sigma>0$, implying $\beta'=0$. Hence in contrast to the DP class $\beta\neq\beta'$ for $d\geqslant 2$. At $d=2$ random walks are barely recurrent, and logarithmic corrections can be found. In this case the generalized hyperscaling law (Mendes *et al.*, 1994) is valid among the exponents

$$2\left(1+\frac{\beta}{\beta'}\right)\delta' + 2\eta' = dz. \quad (155)$$

In $d=1$, however, $\beta=\beta'$ holds owing to an exact duality mapping (Mussawisade *et al.*, 1998) and the hyperscaling is the same as that of the DP class [Eq. (90)].

The RG analysis of BARWe processes for $d<2$ runs into difficulties. These stem from the presence of another critical dimension $d_c'=4/3$ (above which the branching reaction is relevant at $\sigma=0$, and irrelevant for $d<d_c'$). Hence the $d=1$ dimension cannot be accessed by controlled expansions from $d_c=2$. Truncated one-loop expansions (Cardy and Täuber, 1996) for $d=1$ resulted in the exponents

$$\beta=4/7, \quad \nu_{\|}=3/7, \quad \nu_{\perp}=7/17, \quad Z=2, \quad (156)$$

which are quite far from the numerical values determined by Jensen's simulations (Jensen, 1994; Table XVII). Here the cluster exponents δ and η corresponding to a sector with an even number of initial particles are shown. In the case of an odd number of initial particles they exchange values.

It has been conjectured (Deloubrière and van Wijland, 2002) that in one-dimensional fermionic (single-occupancy) and bosonic (multiple-occupancy) models may have different critical behaviors. Since only bosonic field theory exists, which gives rather inaccurate critical exponent estimates, Ódor and Menyhárd (2002) performed bosonic simulations to investigate the density decay of BARW2 from a random initial state. Figure 13 shows the local slopes of the density decay [$\alpha_{\rm eff}$ (93)] around the critical point for several branching rates (σ). The critical point is estimated at $\sigma_c=0.046\,85(5)$, with the corresponding decay exponent $\alpha=0.290(3)$. This value agrees with that of the parity-conserving class.

If there is no explicit diffusion of particles besides the $AA\to\emptyset$, $A\to 3A$ processes (called DBAP by Sudbury, 1990), an implicit diffusion can still be generated by spatially asymmetric branching: $A\emptyset\emptyset\to AAA$ and $\emptyset\emptyset A\to AAA$ from which a diffusion may go on by two lattice steps: $A\emptyset\emptyset\to AAA\to\emptyset\emptyset A$. As a consequence the decay process slows down, and a single particle cannot join a domain, hence domain sizes exhibit parity conser-

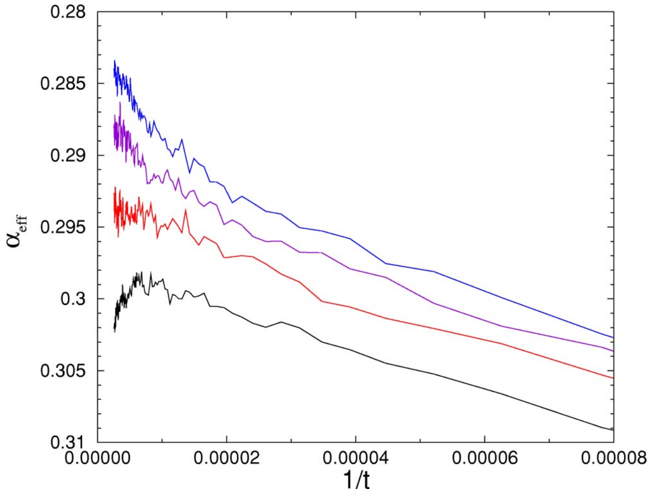

FIG. 13. (Color in online edition) Local slopes [Eq. (93)] of the density decay in a bosonic BARW2 model. Different curves correspond to σ=0.466,0.468,0.469,0.47 (from bottom to top). From Ódor and Menyhárd, 2002.

vation. This results in additional new sectors (besides the existing two BARW2 sectors) that depend on the initial conditions. For example, in the case of random initial distribution $\delta \sim 0.13(1)$ was measured by simulations (Hinrichsen and Ódor, 1999a). Similar sector decomposition has been observed in the diffusion of k-mer models [see, for example, Barma *et al.* (1993) and Barma and Dhar (1994)].

2. The nonequilibrium kinetic Ising model

Another important representative of the parity-conserving class appears among nonequilibrium Ising models, in which the steady state is generated by kinetic processes in connection with heat baths at different temperatures (DeMasi *et al.*, 1985, 1986; Gonzaléz-Miranda *et al.*, 1987; Wang and Lebowitz, 1988; Droz *et al.*, 1989). This research has shown that phase transitions are possible even in one dimension under nonequilibrium conditions (for a review, see Rácz, 1996). In short-ranged interaction models any nonzero-temperature spin-flip dynamics cause a disordered steady state. Menyhárd proposed a class of general nonequilibrium kinetic Ising models (NEKIM) with combined spin-flip dynamics at $T=0$ and Kawasaki spin-exchange dynamics at $T=\infty$, in which, for a range of parameters of the model, a parity-conserving transition takes place (Menyhárd, 1994).

A general form (Glauber, 1963) of the Glauber spin-flip transition rate in one dimension for spin $s_i=\pm 1$ sitting at site i is

$$w_i = \frac{\Gamma}{2}(1+\tilde{\delta}s_{i-1}s_{i+1})\left(1-\frac{\tilde{\gamma}}{2}s_i(s_{i-1}+s_{i+1})\right). \tag{157}$$

Here $\tilde{\gamma}=\tanh(2J/kT)$, J denotes the coupling constant in the ferromagnetic Ising Hamiltonian, and Γ and $\tilde{\delta}$ are further parameters, which can in general also depend on temperature. The Glauber model is a special case corresponding to $\tilde{\delta}=0,\Gamma=1$. There are three independent rates:

$$w_{\uparrow\uparrow\uparrow} = \frac{\Gamma}{2}(1+\tilde{\delta})(1-\tilde{\gamma}), \quad w_{\downarrow\uparrow\downarrow} = \frac{\Gamma}{2}(1+\tilde{\delta})(1+\tilde{\gamma}),$$

$$w_{\uparrow\uparrow\downarrow} = \frac{\Gamma}{2}(1-\tilde{\delta}). \tag{158}$$

In the NEKIM $T=0$ is taken, thus $\tilde{\gamma}=1$, $w_{\uparrow\uparrow\uparrow}=0$, and Γ, $\tilde{\delta}$ are the control parameters to be varied.

The Kawasaki spin-exchange rate of neighboring spins is

$$w_{ii+1}(s_i,s_{i+1}) = \frac{p_{ex}}{2}(1-s_is_{i+1})\left(1-\frac{\tilde{\gamma}}{2}(s_{i-1}s_i+s_{i+1}s_{i+2})\right). \tag{159}$$

At $T=\infty$ ($\tilde{\gamma}=0$) the above exchange is simply an unconditional nearest-neighbor exchange:

$$w_{ii+1} = \frac{1}{2}p_{ex}[1-s_is_{i+1}], \tag{160}$$

where p_{ex} is the probability of spin exchange. The transition probabilities in Eqs. (157) and (160) are responsible for the basic elementary processes of kinks (K). Kinks separating two ferromagnetically ordered domains can carry out random walks with probability

$$p_{rw} \propto 2w_{\uparrow\uparrow\downarrow} = \Gamma(1-\tilde{\delta}), \tag{161}$$

while two kinks getting into neighboring positions will annihilate with probability

$$p_{an} \propto w_{\downarrow\uparrow\downarrow} = \Gamma(1-\tilde{\delta}). \tag{162}$$

Here $w_{\uparrow\uparrow\uparrow}$ is responsible for the creation of kink pairs inside of ordered domains at $T\neq 0$. In the case of spin exchanges, which act only at domain boundaries, the process of main importance here is that a kink can produce two offspring at the next time step with probability

$$p_{K\to 3K} \propto p_{ex}. \tag{163}$$

The above-mentioned three processes compete, and it depends on the values of the parameters Γ, $\tilde{\delta}$, and p_{ex} what the result of this competition will be. It is important to realize that the process $K\to 3K$ can develop into propagation of offspring only if $p_{rw}>p_{an}$, i.e., the new kinks are able to travel to some lattice points away from their place of birth and can thus avoid immediate annihilation. It is seen from the above definitions that $\tilde{\delta}<0$ is necessary for this to happen. In the opposite case the only effect of the $K\to 3K$ process on the usual Ising kinetics is to soften the domain walls. In the development of the NEKIM the normalization condition $p_{rw}+p_{an}+p_{k\to 3k}=1$ was set.

The phase diagram determined by simulations and generalized mean-field calculations (Menyhárd and Ódor, 1995, 2000) is shown in Fig. 14. The line of phase transitions separates two kinds of steady states reach-

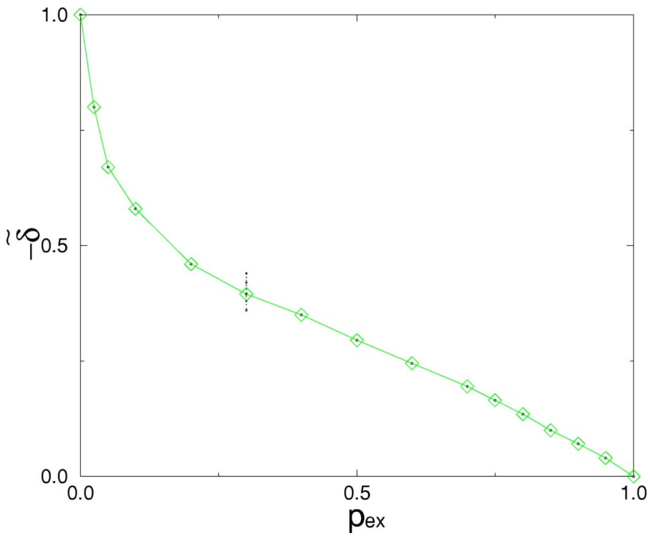

FIG. 14. (Color in online edition) Phase diagram of the two-parameter model. The transversal dotted line indicates the critical point that was investigated in more detail. From Menyhárd and Ódor, 2000.

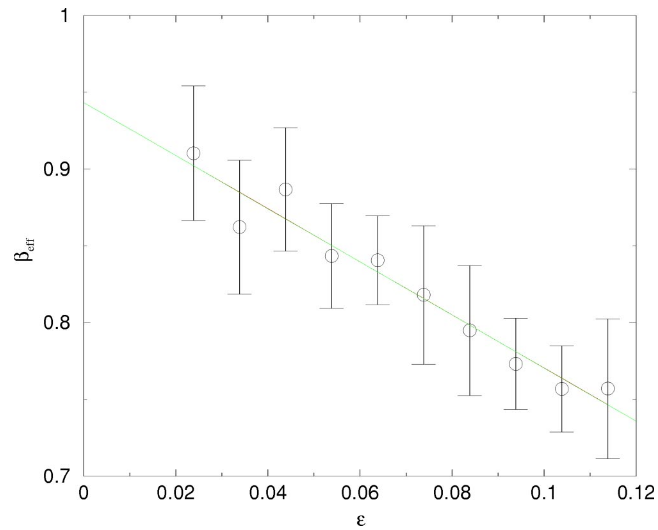

FIG. 15. (Color in online edition) $\beta_{\mathrm{eff}}=d\ln\rho_\infty/d\ln\epsilon$ of kinks (circles) near the critical point ($\epsilon=|\tilde{\delta}-\tilde{\delta}'|$) and linear extrapolation (line) to the asymptotic value [β=0.95(2)]. Simulations were performed on a one-dimensional NEKIM ring of size $L=24\,000$. From Menyhárd and Ódor, 2000.

able by the system for large times: in the Ising phase, supposing that an even number of kinks are present in the initial states, the system orders in one of the possible ferromagnetic states of all spins up or all spins down, while the active phase is disordered from the point of view of the underlying spins. The cause of disorder is the steadily growing number of kinks with time. While the low-level, $N=1,2$ mean-field solutions for the stochastic cellular automaton version of NEKIM exhibit first-order transitions, for $N>2$ this becomes continuous. Generalized mean-field approximations (up to $N=6$) with coherent-anomaly-method extrapolation found $\beta\simeq 1$ (Menyhárd and Ódor, 1995). Recent high-precision Monte Carlo simulations (Menyhárd and Ódor, 2000) resulted in critical exponents $\beta=0.95(2)$ (see Fig. 15) and $\delta=0.280(5)$ at the dotted line of the phase diagram.

Mussawisade *et al.* (1998) have shown that an exact duality mapping exists in the phase diagram of the NEKIM:

$$p'_{an}=p_{an},$$

$$p'_{rw}=p_{an}+2p_{ex},$$

$$2p'_{ex}=p_{rw}-p_{an}. \tag{164}$$

The regions mapped onto each other have the same physical properties. In particular, the line $p_{ex}=0$ maps onto the line $p_{rw}=p_{an}$ and the fast-diffusion limit onto the limit $p_{ex}\to\infty$. There is a self-dual line at

$$\tilde{\delta}=\frac{-2p_{ex}}{1-p_{ex}}. \tag{165}$$

By various static and dynamical simulations, spin and kink density critical exponents have been determined by Menyhárd and Ódor (1996) and as a consequence of the generalized hyperscaling law for the structure factor

$$S(0,t)=L[\langle M^2\rangle-\langle M\rangle^2]\propto t^x, \tag{166}$$

and kink density

$$n(t)=\frac{1}{L}\left\langle \sum_i \frac{1}{2}(1-s_i s_{i+1})\right\rangle \propto t^{-y} \tag{167}$$

the exponent relation

$$2y=x \tag{168}$$

is established. Spins clusters at the parity-conserving point grow by compact domains as in the Glauber point, albeit with different exponents (Menyhárd and Ódor, 1998). The spin-cluster critical exponents in the magnetic field are summarized in Table I. The global persistence (θ_g) and time autocorrelation exponents (λ) were determined both at the Glauber and at the parity-conserving critical points (Menyhárd and Ódor, 1997) and are shown in Table XVIII. While at the Glauber point the scaling relation (17) is satisfied by these exponents, this is not the case at parity-conserving criticality. Therefore the magnetization is a non-Markovian process here. By applying an external magnetic field h that breaks the Z_2 symmetry, one causes the transition type of the model to change to the DP type (see Table XIX; Menyhárd and Ódor, 1996).

Menyhárd and Ódor (2002) introduced a related model in which the symmetry of the spin updates was broken in such a way that two different types of domain walls emerged. The following changes to the Glauber spin-slip rates [Eq. (158)] with $\Gamma=1,\tilde{\delta}=0$ were introduced:

$$w_{\uparrow\downarrow\uparrow}=0, \tag{169}$$

TABLE XVIII. Simulation data for static and dynamic critical spin exponents for the nonequilibrium kinetic ising model (NEKIM).

	β_s	γ_s	$\nu_{\perp,s}$	Z	θ_g	λ_s
Glauber-Ising	0	1/2	1/2	2	1/4	1
Parity-conserving	0.00(1)	0.444(2)	0.444(2)	1.75(1)	0.67(1)	1.50(2)

$$w_{\uparrow\uparrow\downarrow} = w_{\downarrow\uparrow\uparrow} = p_+ < 1/2, \tag{170}$$

while the spin-exchange part remained the same. In the terminology of domain walls as particles, the following reaction-diffusion picture arises. Owing to the symmetry breaking there are two kinds of domain walls $\downarrow\uparrow \equiv A$ and $\uparrow\downarrow \equiv B$, which can only occur alternately $(\ldots B..A..B..A..B\ldots A\ldots)$ owing to the spin background. When the two types meet, $AB \to \emptyset$ happens, while in the opposite sequence, BA, the two domain-walls are repulsive due to Eq. (169). The spin exchange leads to the $A \leftrightarrow ABA$ and $B \leftrightarrow BAB$ type of kink reactions, which together with the diffusion of A's and B's leads to a kind of two-component, coupled branching and annihilating random walk (see Sec. V.K). There are two control parameters in this model: p_{ex}, which regulates the kink production-annihilation, and p_+, which is responsible for local symmetry breaking [Eq. (170)]. Simulations show that for $p_{ex} \to 0$, $p_+ < 0.5$ an absorbing phase emerges with exponents belonging to the N-BARW2 class, a class of even-offspring, N-component BARW, while the transition on the $p_{ex} > 0$ line belongs to the (1+1)-dimensional DP class. (Owing to the pairwise order of kinks, hard-core effects cannot play a role.) Since the $AB \to \emptyset$ reaction breaks the parity conservation of the species, but preserves the global parity conservation, the necessary conditions for the N-BARW2 class can be eased. On the other hand, the occurrence of the DP transition introduces a zero-branching-rate condition for N-BARW2 universal behavior. This study and the results for the generalized contact process (Sec. V.K) emphasize that the conditions for the N-BARW2 class should be further investigated.

A generalization is the probabilistic cellular automaton version of NEKIM, sometimes known as NEKIM-CA, which consists in keeping the spin-flip rates given in Eqs. (158) and prescribing synchronous updating. In this case the $K \to 3K$ branching is generated without the need of an additional, explicit spin-exchange process, and for certain values of parameter pairs $(\Gamma, \tilde{\delta})$ with $\tilde{\delta} < 0$ a parity-conserving transition takes place. The phase boundary of NEKIM-CA in the $(\Gamma, -\tilde{\delta})$ plane is similar to that in Fig. 14 except that the highest value of $\Gamma = 1, \tilde{\delta}_c = 0$ cannot be reached; the limiting value is $\tilde{\delta}_c = -0.065$.

Another possible variant of NEKIM was introduced by Menyhárd and Ódor (2000) in which the Kawasaki rate equation (159) is considered at some finite temperature, instead of $T = \infty$, but keeping $T = 0$ in the Glauber part of the rule. When the temperature is lowered the spin-exchange process acts against the kink production and a parity-conserving class transition occurs. In this case the active phase part of the phase diagram shrinks. For more details see Ódor *et al.* (1999).

The damage-spreading transition of this model coincides with the critical point, and the scaling behavior of spin and kink damage is the same as that of the corresponding NEKIM variables (Ódor and Menyhárd, 1998).

3. Parity-conserving stochastic cellular automata

The first models in which a non-DP-class transition to an absorbing state was firmly established were one-dimensional stochastic cellular automata defined by Grassberger *et al.* (1984). In these models the 00 and 11 pairs follow BARW2 dynamics and their density is the order parameter that vanishes at some critical point. The parity-conserving-class critical exponents were estimated by simulations for these models by Grassberger (1989b). While in the "A" model the critical point coincides with the damage-spreading transition point and both of them are parity-conserving type, in the "B" model the damage-spreading transition occurs in the active phase—where the symmetry of replicas is broken—and therefore the damage-spreading exponents belong to the DP class (Ódor and Menyhárd, 1998).

Another stochastic cellular automaton, which may exhibit a parity-conserving-class transition and which is studied later from different directions, is also introduced here. It points out that the underlying BARW2 dynamics of domain walls in Z_2-symmetric systems can sometimes be seen on the coarse-grained level only. This generalization of the Domany-Kinzel stochastic cellular automa-

TABLE XIX. Coherent-anomaly-method estimates for the kink density and its fluctuation exponents.

h	0.0	0.01	0.05	0.08	0.1	DP
β	1.0	0.281	0.270	0.258	0.285	0.2767(4)
γ		0.674	0.428	0.622	0.551	0.5438(13)

ton (Domany and Kinzel, 1984; see Sec. IV.A.2) was introduced by Hinrichsen (1997). The model has $n+1$ states per site: one active state Ac and n different inactive states $I_1, I_2, \ldots, I_n$. The conditional updating probabilities are given by ($k, l=1, \ldots, n;\ k \neq l$)

$$P(I_k|I_k,I_k) = 1, \tag{171}$$

$$P(Ac|Ac,Ac) = 1 - nP(I_k|Ac,Ac) = q, \tag{172}$$

$$P(Ac|I_k,Ac) = P(Ac|Ac,I_k) = p_k, \tag{173}$$

$$P(I_k|I_k,Ac) = P(I_k|Ac,I_k) = 1 - p_k,$$

$$P(Ac|I_k,I_l) = 1, \tag{174}$$

and the symmetric case $p_1, \ldots, p_n = p$ was explored. Equations (171)–(173) are straightforward generalizations of Eqs. (98)–(100). The only different process is the creation of active sites between two inactive domains of different colors in Eq. (174). For simplicity the probability of this process was chosen to be equal to 1.

For $n=1$ the model defined above reduces to the original Domany-Kinzel model. For $n=2$ it has two Z_2-symmetrical absorbing states. The phase diagram of this model is very similar to that of the Domany-Kinzel model (Fig. 6) except the transition line is of the parity-conserving type. If we call the regions separating inactive domains I_1 and I_2 domain walls (denoted by K), they follow a BARW2 process,

$$K \to 3K, \quad 2K \to \emptyset, \quad K\emptyset \leftrightarrow \emptyset K, \tag{175}$$

but for $1 > q > 0$ the size of the active regions and hence the domain walls remains finite. Therefore observation of the BARW2 process is not so easy and can be done on a coarse-grained level only (except at the end point at $q=0$, where active sites really look like kinks of the NEKIM). Series expansions for the transition point and for the order-parameter critical exponent resulted in $\beta = 1.00(5)$ (Jensen, 1997), which is slightly higher than the most precise simulation results (Menyhárd and Ódor, 2000) but agrees with estimates by the mean-field +coherent-anomaly method (Menyhárd and Ódor, 1995). Similarly to the NEKIM the application of an external symmetry-breaking field changes the transition from the parity-conserving class to the DP class (Hinrichsen, 1997). The other symmetrical end point $\left(q = 1, p = \frac{1}{2}\right)$ in the phase diagram again shows different scaling behavior (here three types of compact domains grow in competition, while the boundaries perform annihilating random walks with exclusions (see. Sec. V.B).

Models with $n > 3$ symmetric absorbing states in one dimension do not show phase transitions (they are always active). In terms of domain walls as particles they are related to ($N > 1$)-component N-BARW2 processes, which exhibit a phase transition only for zero branching rate (see Sec. V.K; Cardy and Täuber, 1996; Hooyberghs *et al.*, 2001).

Generalized Domany-Kinzel-type models—exhibiting n symmetric absorbing states—can be generalized to higher dimensions. In two dimensions Hinrichsen's spreading simulations for the $n=2$ case yielded mean-field-like behavior with $\delta=1$, $\eta=0$, and $z=1$, leading to the conjecture that $1 < d_c < 2$. A similar model exhibiting Potts-like Z_n-symmetric absorbing states in $d=2$ yielded similar spreading exponents, but a first-order transition ($\beta = \alpha = 0$) for $n=2$ (Lipowski and Droz, 2002a). In three dimensions the same model seems to exhibit a mean-field-like transition with $\beta=1$. The verification of these findings would require further research.

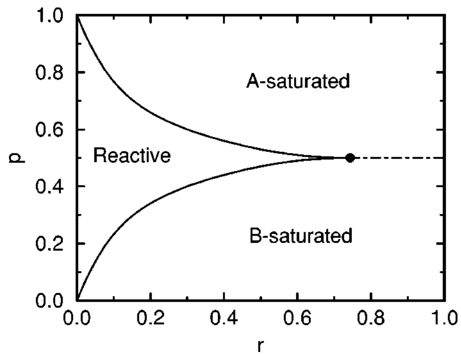

FIG. 16. Phase diagram of the monomer-monomer model. Here p denotes the adsorption probability of A-s, r the reduced adsorption probability of A-s as the consequence of nearest-neighbor repulsion. From Brown *et al.*, 1997.

4. Parity-conserving-class surface-catalytic models

In this subsection I discuss some one-dimensional, surface-catalytic-type reaction-diffusion models exhibiting parity-conserving-class transitions. Strictly speaking they are multicomponent models, but I show that the symmetries among species enable us to interpret the domain-wall dynamics as a simple BARW2 process.

The two-species monomer-monomer (MM) model was first introduced by Zhuo *et al.* (1993). Two monomers, called A and B, adsorb at the vacant sites of a one-dimensional lattice with probabilities p and q, respectively, where $p+q=1$. The adsorption of a monomer at a vacant site is affected by monomers present on neighboring sites. If either neighboring site is occupied by the same species as that trying to adsorb, the adsorption probability is reduced by a factor $r < 1$, mimicking the effect of a nearest-neighbor repulsive interaction. Unlike monomers on adjacent sites react immediately and leave the lattice, leading to a process limited by adsorption only. The basic reactions are

$$\emptyset \to A, \quad \emptyset \to B, \quad AB \to \emptyset. \tag{176}$$

The phase diagram, displayed in Fig. 16 with p plotted vs r, shows a reactive steady state containing vacancies bordered by two equivalent saturated phases (labeled A and B). The transitions from the reactive phase to either of the saturated phases are continuous, while the transition between the saturated phases is first-order discontinuous. The two saturated phases meet the reactive phase at a *bicritical point* at a critical value of $r=r_c$. In

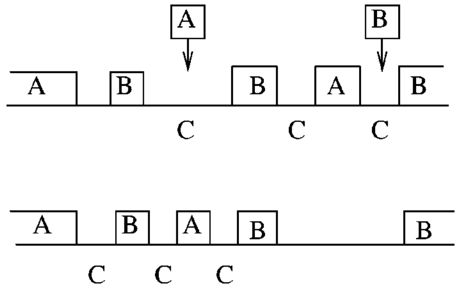

FIG. 17. Domain-wall dynamics in the interacting monomer-monomer model: A particle reaction, branching; B particle reaction, annihilation.

the case of $r=1$, the reactive region no longer exists and the only transition is a first-order discontinuous line between the saturated phases. Considering the density of vacancies between unlike species as the order parameter (which can also be called a species "C"), the model is the so-called "three-species monomer-monomer model." Simulations and cluster mean-field approximations were applied to investigate the phase transitions of these models (Bassler and Browne, 1996, 1997, 1998; Brown *et al.*, 1997). As Fig. 17 shows, if we consider the extended objects filled with vacancies between different species to be domain walls (C), we can observe $C\to 3C$ and $2C \to \emptyset$ BARW2 processes in terms of them. These C parity-conserving processes arise as a combination of the elementary reaction steps (176). The reactions always take place at domain boundaries, hence the Z_2-symmetric A and B saturated phases are absorbing.

The interacting monomer-dimer model (Kim and Park, 1994) is a generalization of the simple monomer-dimer model (Ziff *et al.*, 1986), in which particles of the same species have nearest-neighbor repulsive interactions. This model is parametrized by specifying that a monomer (A) can adsorb at a nearest-neighbor site of an already-adsorbed monomer (restricted vacancy) at a rate $r_A k_A$ with $0 \leqslant r_A \leqslant 1$, where k_A is the adsorption rate of a monomer at a free vacant site with no adjacent monomer-occupied sites. Similarly, a dimer (B_2) can adsorb at a pair of restricted vacancies (B in nearest-neighbor sites) at a rate $r_B k_B$ with $0 \leqslant r_B \leqslant 1$, where k_B is the adsorption rate of a dimer at a pair of free vacancies. There are no nearest-neighbor restrictions in adsorbing particles of different species, and the $AB\to\emptyset$ desorption reaction happens with probability 1. The case $r_A = r_B = 1$ corresponds to the ordinary noninteracting monomer-dimer model, which exhibits a first-order phase transition between two saturated phases in one dimension. In the other limiting case, $r_A = r_B = 0$, there exists no fully saturated phase of monomers or dimers. However, this does not mean that this model no longer has any absorbing states. In fact, there are two equivalent (Z_2-symmetric) absorbing states in this model. These states comprise only the monomers at the odd- or even-numbered lattice sites. A pair of adjacent vacancies is required for a dimer to adsorb, so a state with alternating sites occupied by monomers can be identified with an absorbing state. The parity-conserving-class phase transition of the $r_A = r_B = 0$ infinite repulsive case has been thoroughly investigated (Kim and Park, 1994; Kwon and Park, 1995; Park and Park, 1995; Park *et al.*, 1995; Hwang *et al.*, 1998). As one can see, the basic reactions are similar to those of the MM model [Eq. (176)] but the order parameter here is the density of dimers (K) that may appear between ordered domains of alternating sequences: $0A0..A0A$ and $A0A..0A0$, where monomers are on even or odd sites only. The recognition of an underlying BARW2 process (175) is not so easy in this case. Still, considering regions between odd and even filled ordered domains, one can identify domain-wall random-walk, annihilation, and branching processes through the reactions with dimers, as one can see from the examples below. The introduction of a Z_2-symmetry-breaking field, which makes the system prefer one absorbing state to another, was shown to change that transition type from parity-conserving to DP (Park and Park, 1995):

t	A 0 A 0 A 0 A 0 A 0 0 A 0 A 0 A	K
$t+1$	A 0 A 0 A 0 A 0 A B B A 0 A 0 A	K
$t+2$	A 0 A 0 A 0 A 0 0 0 B A 0 A 0 A	K K K
t	A 0 A 0 A 0 A 0 A 0 0 0 A 0 A 0	K K
$t+1$	A 0 A 0 A 0 A 0 A 0 A 0 A 0 A 0	

5. Nonequilibrium kinetic Ising model with long-range correlated initial conditions

The effect of initially long-range correlations has already been discussed for two-dimensional Ising models (Sec. III.A.5) and for one-dimensional bond percolating systems (Sec. IV.A.7). In both cases continuously changing decay exponents have been found. In case of the NEKIM (see Sec. IV.D.2), simulations (Ódor *et al.*, 1999; Menyhárd and Ódor, 2000) have shown that the density of kinks $\rho_k(t)$ changes as

$$\rho_k(t) \propto t^{\kappa(\sigma)} \tag{177}$$

when the system begins with two-point correlated kink distributions of the form of Eq. (122). The $\kappa(\sigma)$ changes linearly between the two extremes $\beta/\nu_{\|} = \pm 0.285$, as shown in Fig. 18. This behavior is similar to that of the DP model (Sec. IV.A.7), but here one can observe a symmetry:

$$\sigma \leftrightarrow 1-\sigma, \quad \kappa \leftrightarrow -\kappa, \tag{178}$$

which is related to the duality symmetry of the NEKIM [Eq. (164)].

6. The Domany-Kinzel cellular automaton with spatial boundary conditions

The surface critical behavior of the parity-conserving class has been explored through the study of the generalized Domany-Kinzel model (Sec. IV.D.3; Lauritsen *et al.*, 1998; Howard *et al.*, 2000; Frojdh *et al.*, 2001). The basic idea is that on the surface one may include not

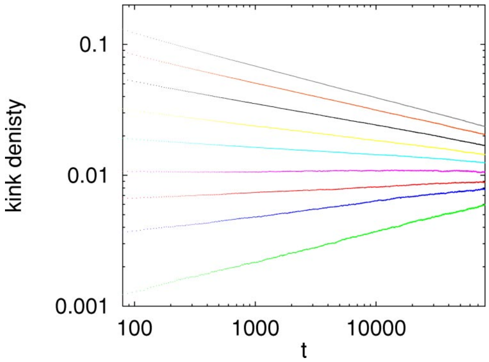

FIG. 18. (Color in online edition) Kink density $\ln[\rho_k(t)]$ vs $\ln(t)$ in NEKIM simulations for $\sigma=0,0.1,0.2,\ldots,1$ initial conditions (from bottom to top curves). From Menyhárd and Ódor, 2000.

only the usual BARW2 reactions [Eq. (175)] but also a parity symmetry-breaking $A\to\varnothing$ reaction. Depending on whether or not the $A\to\varnothing$ reaction is actually present, we may then expect different boundary universality classes. Since time-reversal symmetry [Eq. (88)] is broken for BARW2 processes, two independent exponents ($\beta_{1,\mathrm{seed}}$, $\beta_{1,\mathrm{dens}}$) characterize the surface critical behavior.

The surface phase diagram for the mean-field theory of BARW (valid for $d>d_c=2$) is shown in Fig. 19. Here σ_m, σ_{m_s} are the rates for the branching processes $A\to(m+1)A$ in the bulk and at the surface, respectively, and μ_s is the rate for the surface spontaneous annihilation reaction $A\to\varnothing$. Otherwise, the labeling is the same as that for the DP phase diagram (see Fig. 7). The $\mu_s>0$ corresponds to the parity symmetry-breaking reflecting boundary condition.

For the $\mu_s=0$ (inactive boundary condition) parity-conserving case, the surface action is of the form

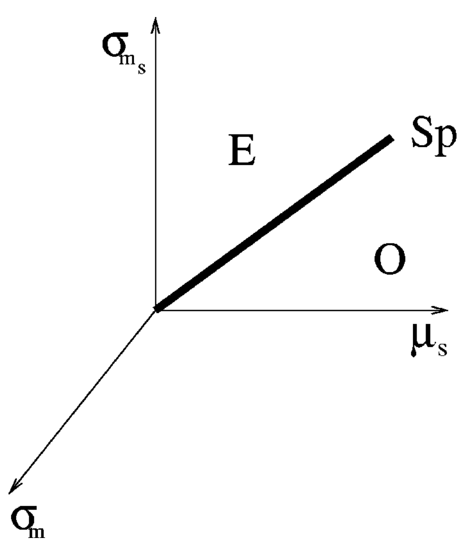

FIG. 19. Schematic mean-field boundary phase diagram for BARW. See text for an explanation of the labeling. From Fröjdh *et al.*, 2001.

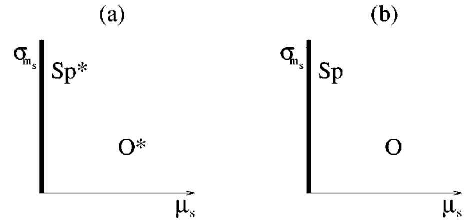

FIG. 20. Schematic surface phase diagrams for BARW in $d=1$ for (a) $\sigma_m<\sigma_{m,\mathrm{critical}}$, and (b) $\sigma_m=\sigma_{m,\mathrm{critical}}$. See text for an explanation of the labeling. From Fröjdh *et al.*, 2001.

$$S_s=\int d^{d-1}x_{\parallel}\int_0^{\tau}dt\sum_{l=1}^{m/2}\sigma_{2l_s}(1-\psi_s^{2l}\psi_s\phi_s), \tag{179}$$

where $\psi_s=\psi(\mathbf{x}_{\parallel},x_{\perp}=0,t)$ and $\phi_s=\phi(\mathbf{x}_{\parallel},x_{\perp}=0,t)$. In $d=1$ the boundary and bulk transitions are inaccessible to controlled perturbative expansions, but scaling analysis shows that surface branching is irrelevant, leading to the $Sp*$ and Sp special transitions. For the case of $\mu_s>0$ (reflecting boundary condition) parity symmetry breaking, the surface action is

$$S_2=\int d^{d-1}x_{\parallel}\int_0^{\tau}dt\Bigg(\sum_{l=1}^{m}\sigma_{l_s}(1-\psi_s^{l})\psi_s\phi_s + \mu_s(\psi_s-1)\phi_s\Bigg), \tag{180}$$

and the RG procedure shows that the stable fixed point corresponds to an ordinary transition. Therefore in one dimension the phase diagram looks very different from the mean-field case (Fig. 20). One can differentiate two cases corresponding to (a) the annihilation fixed point of the bulk and (b) the parity-conserving critical point of the bulk. As one can see in both cases the ordinary transition (O,O^*) corresponds to $\mu_s>0$, the reflecting boundary condition, and the special transitions (Sp,Sp^*) to $\mu_s=0$, the inactive boundary condition. The active boundary condition obviously behaves as if there existed a surface reaction equivalent to $\varnothing\to A$, and thus it belongs to the normal transition universality class. By scaling considerations the following scaling relations can be derived:

$$\tau_1=\nu_{\parallel}-\beta_{1,\mathrm{dens}}, \tag{181}$$

$$\nu_{\parallel}+d\nu_{\perp}=\beta_{1,\mathrm{seed}}+\beta_{\mathrm{dens}}+\gamma_1. \tag{182}$$

Howard *et al.* (2000) showed that on the self-dual line of the one-dimensional BARWe model (see Sec. IV.D.2 and Mussawisade *et al.*, 1998) the scaling relations between exponents of ordinary and special transitions,

$$\beta_{1,\mathrm{seed}}^{\mathrm{O}}=\beta_{1,\mathrm{dens}}^{\mathrm{Sp}} \tag{183}$$

and

$$\beta_{1,\mathrm{seed}}^{\mathrm{Sp}}=\beta_{1,\mathrm{dens}}^{\mathrm{O}}, \tag{184}$$

hold. Relying on universality Howard *et al.* claim that they should be valid elsewhere close to the transition

TABLE XX. Critical boundary exponents of the parity-conserving class in $d=1,2$ for ordinary and special cases: IBC, inactive boundary condition; RBC, reflecting boundary condition; O, one dimensional; Sp, special.

	One dimensional (IBC)	One dimensional (RBC)	Two dimensional (O)	Two dimensional (Sp)
$\beta_{1,\mathrm{seed}}$	2.06(2)	1.37(2)	0	0
$\beta_{1,\mathrm{dens}}$	1.34(2)	2.04(2)	3/2	1
τ_1	1.16(4)	1.85(4)	1	1
γ_1	2.08(4)	2.77(4)	1/2	1/2

line. Numerical simulations support this hypothesis, as shown in Table XX.

E. Branching with $kA \to \varnothing$ annihilation

In Sec. IV.C.1 the $2A \to \varnothing$ annihilating random walk (ARW) was introduced. By adding branching processes to it we defined the BARWo and BARWe models exhibiting continuous phase transitions (see Secs. IV.A.3 and IV.D.1). Now we generalize this construction to models with m branching and $kA \to \varnothing$ annihilation (BkARW), formulating the field-theoretical action

$$S = \int d^d x dt [\psi(\partial_t - D\nabla^2)\phi - \lambda(1-\psi^k)\phi^k + \sigma(1-\psi^m)\psi\phi]. \tag{185}$$

These systems for $k>2$ exhibit continuous phase transitions at zero branching rate with an upper critical dimension: $d_c = 2/(k-1)$ (Lee, 1994; Cardy and Täuber, 1998) with the mean-field exponents

$$\alpha = \beta = 1/(k-1), \quad Z=2, \quad \nu_\perp = 1/2. \tag{186}$$

At d_c (which falls below physical dimensions for $k>3$) the decay has logarithmic corrections:

$$\rho(t) = A_k[\ln(t)/t]^{1/(k-1)}. \tag{187}$$

So for the $AAA \to \varnothing$ process in one dimension this gives the decay behavior (Lee, 1994)

$$\rho(t) = \left(\frac{1}{4\pi\sqrt{3}D}\right)^{1/2}[\ln(t)/t]^{1/2} + O(t^{1/2}), \tag{188}$$

which is the dominant behavior of the one-dimensional bosonic pair-contact model with particle diffusion at the transition point (see Sec. V.F). Note that for $k=3$, $m=1,2$ the field theory of Cardy and Täuber (1998) predicts DP-class transitions in $d=1$ owing to the BARWo terms generated by renormalization.

F. General $nA \to (n+k)A$, $mA \to (m-l)A$ processes

For a long time reaction-diffusion models with only single-parent branching have been investigated, although there was an early forgotten numerical result by Grassberger (1982c) claiming a non-DP-type of continuous phase transition in a model where particle production can occur by the reaction of two parents. Later it turned out that in such binary production lattice models, where solitary particles follow a random walk (and hence behave like a coupled system), different universal behavior emerges (see Sec. V.F). In this subsection I shall discuss the mean-field classes in the models

$$nA \xrightarrow{\sigma} (n+k)A, \quad mA \xrightarrow{\lambda} (m-l)A, \tag{189}$$

with $n>1$, $m>1$, $k>0$, $l>0$, and $m-l \geqslant 0$. If n particles collide, then they can spawn k additional particles, while if m particles collide, then m of those particles is annihilated. In low dimensions the site-restricted and bosonic versions of these models exhibit different behavior. The field theory of the $n=m=l=2$, bosonic model was investigated by Howard and Täuber (1997), who concluded that it possessed a non-DP-type of criticality with $d_c=2$ (see Sec. V.F). For other cases no rigorous field-theoretical treatments exist. Numerical simulations in one and two dimensions for various $n=3$ and $n=4$ models have resulted in somewhat contradictory results (Park *et al.*, 2002; Kockelkoren and Chaté, 2003a; Ódor, 2003a). There is disagreement as to the value of the upper critical dimension, but in any case d_c seems to be very low ($d_c = 1-2$), hence the number of non-mean-field classes in such models is limited. By contrast, there is a series of mean-field classes depending on n and m.

1. The $n=m$ symmetric case

In such models there is a continuous phase transition at finite production probability: at $\sigma_c > 0$ characterized by the order-parameter exponents (Park *et al.*, 2002; Ódor, 2003a):

$$\beta = 1, \quad \alpha = 1/n. \tag{190}$$

These classes generalize the mean-field class of the directed-percolation (Sec. IV.A) and binary production models (Sec. V.F). For purposes of referencing it in Table XXXII of the Summary, I call it the PARW's (symmetric production and m-particle annihilating random walk) class.

2. The $n>m$ case

In this case the mean-field solution provides a first-order transition (see see Ódor, 2003a), hence it does not imply anything with respect to possible classes for models below the critical dimension ($d<d_c$). Note, however, that by higher-order cluster mean-field approximations, when the diffusion plays a role the transition may become a continuous one (see, for example, Ódor *et al.*, 1993; Menyhárd and Ódor, 1995; Ódor and Szolnoki, 1996).

3. The $n<m$ case

In this case the critical point is at zero production probability $\sigma_c = 0$ where the density decays as α^{MF}

$=1/(m-1)$ as in case of the $n=1$ branching and $m=l$ annihilating models (BkARW classes; see Sec. V.E), but the steady-state exponent is different: $\beta^{MF}=1/(m-n)$, defining another series of mean-field classes (PARWa; Ódor, 2003a). Again cluster mean-field approximations may predict the appearance of other $\sigma>0$ transitions with different critical behavior.

V. UNIVERSALITY CLASSES OF MULTICOMPONENT SYSTEMS

First I shall recall some well-known results (see references in Privman, 1996) for multicomponent reaction-diffusion systems without particle creation. From the viewpoint of phase transitions these describe the behavior in the inactive phase or, in the case of some N-component BARW models, right at the critical point. Then I shall show the effect of particle exclusions in one dimension. Later I shall discuss universal behavior of more complex, coupled multicomponent systems. This field is quite new and some of the results are still under debate.

A. The $A+B\to\emptyset$ classes

The simplest two-component reaction-diffusion model involves two types of particles undergoing diffusive random walks and reacting upon contact to form an inert particle. The action of this model is

$$S=\int d^dxdt[\psi_A(\partial_t-D_A\nabla^2)\phi_A\psi_B(\partial_t-D_B\nabla^2)\phi_B - \lambda(1-\psi_A\psi_B)\phi_A\phi_B]-\rho(0)[\psi_A(0)+\psi_B(0)], \quad (191)$$

where D_A and D_B denote the diffusion constants of species A and B. In $d<d_c=4$ dimensions and for homogeneous, initially equal density of A and B particles (ρ_0) the density decays asymptotically as (Burlatskii and Ovchinnikov, 1978; Ovchinnikov and Zel'dovich, 1978; Toussaint and Wilczek, 1983; Kang and Redner, 1985; Bramson and Lebowitz, 1988; Lee and Cardy, 1995)

$$\rho_A(t)=\rho_B(t)\propto C\sqrt{\Delta}t^{-d/4}, \quad (192)$$

where $\Delta=\rho(0)-C'\rho^{d/2}(0)+\cdots$, C is a universal constant, and C' is a nonuniversal constant. This slow decay behavior is due to the fact that, in the course of the reaction, local fluctuations in the initial distribution of reactants lead to the formation of clusters of like particles that do not react, and they will be asymptotically segregated for $d<4$. The asymptotically dominant process is the diffusive decay of the fluctuations of the initial conditions. Since this is a short-ranged process, the system has a long-time memory—appearing in the amplitude dependence—for the initial density $\rho(0)$. For $d\leqslant 2$ a controlled RG calculation is not possible, but the result (192) gives the leading-order term in $\epsilon=2-d$ expansion. For the case of $D_A\neq D_B$ a RG study (Lee and Cardy, 1995) found a new amplitude but the same exponents.

The persistence behavior in one dimension with homogeneous, equal initial density of particles $[\rho_0=\rho_A(0)+\rho_B(0)]$ was studied by O'Donoghue and Bray (2001). The probability $p(t)$ that an annihilation process has not occurred at a given site (*type-I persistence*) has the asymptotic form

$$p(t)\sim \text{const}+t^{-\theta_l}. \quad (193)$$

For a density of particles $\rho\gg 1$, θ_l is identical to that governing the persistence properties of the one-dimensional diffusion equation, where $\theta_l\approx 0.1207$. In the case of an initially low density, $\rho_0\ll 1$, $\theta_l\approx 1/4$ was found asymptotically. The probability that a site remains unvisited by any random walker (*type-II persistence*) decays in a stretched exponential way:

$$p(t)\sim \exp(-\text{const}\times\rho_0^{1/2}t^{1/4}) \quad (194)$$

provided $\rho_0\ll 1$.

B. $AA\to\emptyset$, $BB\to\emptyset$ with hard-core repulsion

The next-simplest two-component model in which particle blocking may be effective in low dimensions was investigated first in the context of a stochastic cellular automaton model. At the symmetric point of the generalized Domany-Kinzel model (see Sec. IV.D.3), compact domains of $I1$ and $I2$ grow separated by $A=Ac-I1$ and $B=Ac-I2$ kinks that cannot penetrate each other. In particle language, this system is a reaction-diffusion model of two types $A+A\to\emptyset$, $B+B\to\emptyset$ with the exclusion $AB\leftrightarrow BA$ and special pairwise initial conditions (because the domains are bounded by kinks of the same type):

. . . . A . . . A . . . B . B . . B . . . . . B . . A . . A . .

In the case of homogeneous, pairwise initial conditions, simulations by Ódor and Menyhárd (2000) showed a density decay of kinks $\rho\propto t^{-\alpha}$ characterized by a power law with an exponent somewhat larger than $\alpha=0.5$. The $\alpha=1/2$ would have been expected in the case of two copies of ARW systems that do not exclude each other. Furthermore, the deviation of α from 1/2 showed an initial density dependence. Ódor and Menyhárd (2000) provided a possible explanation based on permutation symmetry between types, according to which hard-core interactions cause *marginal perturbation* resulting in nonuniversal scaling. The situation is similar to that of compact directed percolation that is confined by parabolic boundary conditions (see Sec. IV.C.3) if we assume that AB and BA pairs exert parabolic space-time confinement on coarsening domains. Nonuniversal scaling can also be observed at surface critical phenomena. Similarly here, AB, BA pairs produce "multisurfaces" in the bulk. However, simulations and independent interval approximations in a similar model predict logarithmic corrections to the single-component decay with the form $\rho\sim t^{-1/2}/\ln(t)$ (Majumdar *et al.*, 2001). Note that both kinds of behavior may occur in case of marginal perturbations. Cluster simulations (Ódor and Menyhárd, 2000) also showed an initial-density- $[\rho_{I1}(0)]$ dependent survival probability of $I2$'s in the sea of $I1$'s:

$$P_{I2}(t) \propto t^{-\delta[\rho_{I1}(0)]}. \tag{195}$$

However, this reaction-diffusion model with homogeneous, random initial distribution of A's and B's exhibits a much slower density decay. An exact duality mapping helps to understand the coarsening behavior. Consider the left-most particle, which may be either A or B, and arbitrarily relabel it as a particle of species X. For the second particle, we relabel it as Y if it is the same species as the initial particle; otherwise we relabel the second particle as X. We continue to relabel each subsequent particle according to this prescription until all particles are relabeled from $\{A,B\}$ to $\{X,Y\}$. For example, the string

$$AABABBBA\cdots$$

translates to

$$XYYYYXYY\cdots.$$

The diffusion of the original A and B particles at equal rates translates into diffusion of the X and Y particles. Furthermore, the parallel single-species reactions, $A+A\to\emptyset$ and $B+B\to\emptyset$, translate directly to two-species annihilation $X+Y\to\emptyset$ (see Sec. V.A) in the second system. The interesting point is that, in the $X+Y\to\emptyset$ model, blockades do not exist, because XY pairs annihilate, and there is no blockade between XX and YY pairs. Therefore the density decay should be proportional to $t^{-1/4}$. Simulations confirmed this for the $A+A\to\emptyset$, $B+B\to\emptyset$ model (Ódor, 2001c). Nevertheless corrections to scaling were also observed. The pairwise initial condition transforms in the second system to domains of $..XYXY..$ separated by YY and XX pairs, which do not allow X and Y particles to escape each other.

C. Multispecies $A_i+A_j\to\emptyset$ classes

When the diffusion-limited reactions $AB\to\emptyset$ (Sec. V.A) for $q>2$ species are generalized,

$$A_i+A_j\to\emptyset, \tag{196}$$

in $d\geqslant 2$ dimensions the asymptotic density decay for such mutual annihilation processes with equal rates and initial densities is the same as for single-species pair annihilation $AA\to\emptyset$.

In $d=1$, however, particles of different types cannot pass each other and segregation occurs for all $q<\infty$. The total density decays according to a q-dependent power law, $\rho\propto t^{-\alpha(q)}$, with

$$\alpha=(q-1)/2q \tag{197}$$

exactly (Deloubrière *et al.*, 2002). These findings were also supported by Monte Carlo simulations. Special initial conditions such as $\ldots ABCDABCD\ldots$ prevent the segregation and lead to decay of the $2A\to\emptyset$ model (Sec. IV.C.1).

D. Unidirectionally coupled ARW classes

Above we have considered symmetrically coupled ARW systems. Now let us turn to unidirectionally coupled ARW (Sec. IV.C.1) models:

$$\begin{aligned}&A+A\to\emptyset, \quad A\to A+B,\\ &B+B\to\emptyset, \quad B\to B+C,\\ &C+C\to\emptyset, \quad C\to C+D,\\ &\ldots.\end{aligned} \tag{198}$$

These models were introduced and analyzed with the RG technique and simulations by Goldschmidt (1998). In unidirectional coupling, $A\to B$ constitutes a spontaneous death process of A particles leading to exponential density decay. On the other hand, quadratic coupling of the form $A+A\to B+B$ leads to asymptotically decoupled systems (Howard and Täuber, 1997). The mean-field theory is described by the rate equation for the density $\rho_i(x,t)$ at level i:

$$\frac{\partial\rho_i(x,t)}{\partial t}=D\nabla^2\rho_i(x,t)-2\lambda_i\rho_i(x,t)^2+\sigma_{i,i-1}\rho_{i-1}(x,t), \tag{199}$$

which relates the long-time behavior of level i to that of level $i-1$:

$$\rho_i(t)\propto\rho_{i-1}^{1/2}. \tag{200}$$

By inserting into this the exact solution for annihilating random walks (Sec. IV.C.1), one gets

$$\rho_i(t)\sim\begin{cases}t^{-d/2^i} & \text{for } d<2,\\ (t^{-1}\ln t)^{1/2^{i-1}} & \text{for } d=d_c=2,\\ t^{-1/2^{i-1}} & \text{for } d>2.\end{cases} \tag{201}$$

The action of a two-component system with fields a, $\hat{a}$, b, $\hat{b}$ for equal annihilation rates (λ) takes the form

$$S=\int d^dx\int dt[\hat{a}(\partial_t-D\nabla^2)a-\lambda(1-\hat{a}^2)a^2 + \hat{b}(\partial_t-D\nabla^2)b-\lambda(1-\hat{b}^2)b^2+\sigma(1-\hat{b})\hat{a}a]. \tag{202}$$

The RG solution is plagued by IR-divergent diagrams similarly to that of unidirectionally coupled directed percolation (see Sec. VI.H), which can be interpreted as evidence of an eventual nonuniversal crossover to the decoupled regime. Simulation results—exhibiting finite particle numbers and coupling strengths—really show the breakdown of scaling [Eq. (201)], but the asymptotic behavior could not be determined. Therefore the results (201) are valid for an intermediate time region.

E. Directed percolation coupled to frozen fields

One of the first generalizations of absorbing phase-transition models were systems in which percolation is coupled to frozen fields; these classes exhibit many absorbing states, hence not fulfilling the conditions of the DP hypothesis (Janssen, 1981; Grassberger, 1982a; Sec. IV). Several variants of models with infinitely many absorbing states containing frozen particle configurations have been introduced. In these models, nondiffusive (slave) particles are coupled to a DP-like (order parameter) process. In the case of homogeneous, uncorrelated initial conditions, DP-class exponents have been found, whereas in cluster simulations—which involve a correlated initial state of the order-parameter particles—initial-density-dependent scaling exponents (η and δ) arise. These cluster exponents take the DP-class values only if the initial density of the slave particles agrees with the "natural density" that occurs in the steady state. The first such models, introduced by Jensen (1993a) and Jensen and Dickman (1993a) were the so-called pair-contact process (see Sec. V.E) and the dimer reaction model. These systems seem to be single-component ones, when rules for the pairs are defined, but the isolated, frozen particles behave as a second component. In the *threshold transfer process* (Mendes *et al.*, 1994) the two components are defined as the 2's, which follow the DP process, and the 1's, which decay or reappear as $\emptyset \xrightarrow{r} 1 \xrightarrow{1-r} \emptyset$.

For the pair-contact model defined by the simple processes (205), a set of coupled Langevin equations were set up (Muñoz *et al.*, 1996, 1998) for the fields $n_1(\mathbf{x},t)$ and $n_2(\mathbf{x},t)$:

$$\frac{\partial n_2}{\partial t} = [r_2 + D_2\nabla^2 - u_2 n_2 - w_2 n_1]n_2 + \sqrt{n_2}\,\eta_2, \tag{203a}$$

$$\frac{\partial n_1}{\partial t} = [r_1 + D_1\nabla^2 - u_1 n_2 - w_1 n_1] + \sqrt{n_2}\,\eta_1, \tag{203b}$$

where D_i, r_i, u_i, and w_i are constants and $\eta_1(\mathbf{x},t)$ and $\eta_2(\mathbf{x},t)$ are Gaussian white noise terms. Owing to the multiple absorbing states and the lack of the time-reversal symmetry [Eq. (88)] a generalized hyperscaling law (155) was derived by Mendes *et al.* (1994). As discussed by Muñoz *et al.* (1998) this set of equations can be simplified by dropping the D_1, u_1, and noise terms in Eq. (203b) and then solving that equation for n_1 in terms of n_2. Substituting this result for n_1 into Eq. (203a) yields

$$\begin{aligned}\frac{\partial n_2(\mathbf{x},t)}{\partial t} &= D_2\nabla^2 n_2(\mathbf{x},t) + m_2 n_2(\mathbf{x},t) - u_2 n_2^2(\mathbf{x},t)\\ &\quad + w_2[r_1/w_1 - n_1(\mathbf{x},0)]n_2(\mathbf{x},t)e^{-w_1\int_0^t n_2(\mathbf{x},s)ds}\\ &\quad + \sqrt{n_2(\mathbf{x},t)}\,\eta_2(\mathbf{x},t),\end{aligned} \tag{204}$$

where $n_1(\mathbf{x},0)$ is the initial condition of the n_1 field, and $m_2 = r_2 - w_2 r_1/w_1$. The "natural density" (Jensen and Dickman, 1993), n_1^{nat}, then corresponds to the uniform density, $n_1(t=0) = r_1/w_1$, for which the coefficient of the exponential term vanishes, and we get back the Langevin equation (84) for directed percolation. This derivation provides a simple explanation for the numerical observation of DP exponents in the case of natural initial conditions. However, it does not take into account the long-time memory and the fluctuations of passive particles [with power-law time and p dependences (Ódor *et al.*, 1998)]. Therefore some of the terms omitted from this derivation (for instance, the term proportional to n_2^2 in the equation for n_1) cannot be safely eliminated (Marques *et al.*, 2001) and this simplified theory does not generate critical fluctuations for its background field. This treatment has still not provided theoretical proof for the initial-density-dependent spreading exponents observed in simulations (Jensen, 1993a; Jensen and Dickman, 1993; Mendes *et al.*, 1994; Odor *et al.*, 1998) and in the numerical integration of the Langevin equation (López and Muñoz, 1997). Furthermore, the situation is much more complicated when approaching criticality from the inactive phase. In particular, the scaling behavior of n_1 in this case seems to be unrelated to n_2 (this is similar to the diffusive slave field case; see Ódor *et al.*, 2002; Sec. V.F.3). In this case it is more difficult to analyze the field theory and dynamical-percolation-type terms are generated that can be observed in two dimensions by simulations and by mean-field analysis (Muñoz *et al.*, 1996, 1998; Wijland, 2002). Very recently it was claimed, based on the field-theoretical analysis of the generalized epidemic process (Sec. IV.B)—which exhibits similar long-time memory terms—that the cluster variables should follow stretched exponential decay behavior (Jimenez-Dalmaroni and Hinrichsen, 2003).

In two dimensions the critical point of spreading (p_s) moves (as a function of initial conditions) and does not necessarily coincide with the bulk critical point (p_c). The spreading behavior depends on the coefficient of the exponential, non-Markovian term of Eq. (204). For a positive coefficient the p_s falls in the inactive phase of the bulk and the spreading follows dynamical percolation (see Sec. IV.B). For a negative coefficient the p_s falls in the active phase of the bulk, and spreading exponents are nonuniversal (as in one dimension) but satisfy the hyperscaling law [Eq. (155)].

Up to now I have discussed spreading processes with unary particle production. Now I introduce a family of systems with binary particle production (i.e., for a new particle to be produced, two particles need to collide). The pair-contact model is defined on the lattice by the following processes:

$$2A \xrightarrow{1-p} 3A, \quad 2A \xrightarrow{p} \emptyset, \tag{205}$$

such that reactions take place at nearest-neighbor sites and we allow single-particle occupancy at most. The order parameter is the density of nearest-neighbor pairs ρ_2. The pair-contact process exhibits an active phase for

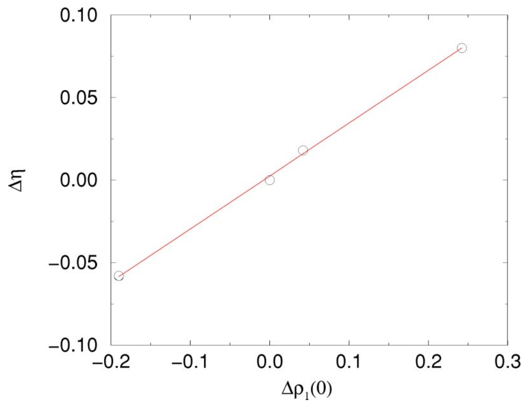

FIG. 21. (Color in online edition) Initial concentration dependence of the exponent η for the pair-contact model. Linear regression gives a slope 0.320(7) between $\eta-\eta_{DP}$ and $\rho_1(0)-\rho_1^{\rm nat}$. From Ódor *et al.*, 1998.

$p<p_c$; for $p\geqslant p_c$ the system eventually falls into an absorbing configuration devoid of nearest-neighbor pairs (but with a density ρ_1 of isolated particles). The best estimate for the critical parameter in one dimension is $p_c=0.077\,090(5)$ (Dickman and de Silva, 1998a, 1998b). Static and dynamic exponents that correspond to initially uncorrelated homogeneous states agree well with those of (1+1)-dimensional directed percolation (Table XII). Spreading exponents that involve averaging over all runs, hence involving the survival probability, are nonuniversal (see Fig. 21; Ódor *et al.*, 1998). The anomalous critical spreading in this model can be traced to a long memory in the dynamics of the order parameter ρ_2, arising from a coupling to an auxiliary field (the local particle density ρ), which remains frozen in regions where $\rho_2=0$. Ódor *et al.* (1998) observed a slight variation of the spreading critical point as a function of $\rho_1(0)$ (similarly to the two-dimensional case), but more detailed simulations (Dickman, 1999) suggest that this could be explained by strong corrections to scaling. Simulations provided numerical evidence that the ρ_1 exhibits anomalous scaling as

$$|\rho_1^{nat}-\rho_1|\propto|p-p_c|^{\beta_1}$$

to $\rho_1^{nat}=0.242(1)$ with a DP exponent (Ódor *et al.*, 1998) for $p<p_c$ and with $\beta_1=0.9(1)$ for $p>p_c$ (Marques *et al.*, 2001). The damage-spreading transition point and the DS exponents of this model coincide with the critical point and the critical exponents of the pair-contact model (Ódor *et al.*, 1998).

The effect of an external particle source that creates isolated particles, and hence does not couple to the order parameter, was investigated by simulations and by mean-field plus coherent anomaly approximations (Dickman, Rabelo, and Ódor, 2001). While the critical point p_c showed a singular dependence on the source intensity, the critical exponents appeared to be unaffected by the presence of the source, except possibly for a small change in β.

The properties of the two-dimensional pair contact process in homogeneous, uncorrelated initial conditions was investigated by simulations (Kamphorst *et al.*, 1999). In this case all six nearest neighbors of a pair were considered for reactions (205). The critical point was located at $p_c=0.2005(2)$. By determining α, $\beta/\nu_\perp$, and Z exponents and order-parameter moment ratios by simulations, Kamphorst *et al.* confirmed the universal behavior of the (2+1)-dimensional DP class (Table XII). The spreading exponents are expected to behave as described in Sec. V.E.

F. Directed percolation coupled to diffusive fields

The next question one can pose following Sec. V.E is whether a diffusive field coupled to a DP process is relevant. Prominent representatives of such models are binary particle production systems with explicit diffusion of solitary particles. The critical behavior of such systems is still under investigation. The annihilation-fission process is defined as

$$2A\xrightarrow{\lambda}\emptyset,\quad 2A\xrightarrow{\sigma}(n+2)A,\quad A\emptyset\overset{D}{\leftrightarrow}\emptyset A. \tag{206}$$

The corresponding action for bosonic particles was derived from the master equation by Howard and Täuber (1997),

$$S=\int d^dxdt[\psi(\partial_t-D\nabla^2)\phi-\lambda(1-\psi^2)\phi^2 + \sigma(1-\psi^n)\psi^2\phi^2]. \tag{207}$$

Usually bosonic field theories describe well the critical behavior of "fermionic" systems (i.e., those with a maximum of one particle per site occupation). This is due to the fact that at absorbing phase transitions the occupation number vanishes. In this case, however, the active phases of bosonic and fermionic models differ significantly: in the bosonic model the particle density diverges, while in the fermionic model there is a steady state with finite density. As a consequence the bosonic field theory cannot describe the active phase or the critical behavior of a fermionic particle system.

As one can see, this theory lacks interaction terms linear in the (massless) field variable ϕ in contrast with the DP action (87). Although the field theory of bosonic annihilation-fission has turned out to be nonrenormalizible, Howard and Täuber (1997) concluded that its critical behavior cannot be in the DP class. In fact, the upper critical behavior is $d_c=2$, which is different from that of the DP and parity-conserving (Ódor *et al.*, 2002). In the Langevin formulation

$$\frac{\partial\rho(x,t)}{\partial t}=D\nabla^2\rho(x,t)+(n\sigma-2\lambda)\rho^2(x,t)+\rho(x,t)\eta(x,t) \tag{208}$$

the noise is complex:

$$\langle\eta(x,t)\rangle=0, \tag{209}$$

$$\langle \eta(x,t)\eta(x',t')\rangle = [n(n+3)\sigma - 2\lambda]\delta^d(x-x')(t-t'),$$

a new feature (it is real in the case of directed percolation and purely imaginery in the case of compact DP and parity-conserving classes).

Equation (208) without noise gives the mean-field behavior of the bosonic model. For $n\sigma > 2\lambda$ the density diverges, while for $n\sigma > 2\lambda$ it decays with a power law, with $\alpha_b^{MF}=1$. The mean-field description in the inactive phase of the bosonic model was found to be valid for the two-dimensional fermionic annihilation-fission system too (Ódor *et al.*, 2002). Here the pair density decays as $\rho_2(t)\propto t^{-2}$ (Ódor *et al.*, 2002) in agreement with the mean-field approximation. Contrary to this, for $\lambda \leqslant \lambda_c$, ρ and ρ_2 seem to be related by a logarithmic ratio $\rho(T)/\rho_2(t) \propto \ln(t)$. This behavior could not be described by the mean-field approximations and possibly due to the $d = d_c = 2$ spatial dimension.

The field theory suggests that the scaling behavior of the one-dimensional bosonic model in the inactive phase is dominated by the $2A\to\varnothing$ process, that has an upper critical dimension $d_c=2$, hence the particle density decays with a power-law exponent: $\alpha=1/2$ (see Sec. IV.E). At the transition point the dynamical behavior is unexplored, but the particle density is expected to tend to constant. This behavior has been confirmed by simulations in case of the one-dimensional annihilation-fission model (Ódor and Menyhárd, 2002).

The field-theoretical description of the fermionic annihilation-fission process runs into even more serious difficulties (Täuber, 2002) than that of the bosonic model and predicts an upper critical dimension $d_c=1$ that contradicts simulation results (Ódor *et al.*, 2002). For the fermionic annihilation-fission system, mean-field approximations (Carlon *et al.*, 2001; Ódor *et al.*, 2002) give a continuous transition with exponents

$$\beta = 1, \quad \beta' = 0, \quad Z = 2, \quad \nu_{\|} = 2, \quad \alpha = 1/2, \quad \eta = 0. \tag{210}$$

These mean-field exponents are distinct from those of other well-known classes, including directed percolation, parity-conserving, and voter model classes. They were confirmed in a two-dimensional fermionic annihilation-fission model, with logarithmic corrections, indicating $d_c=2$ (Ódor *et al.*, 2002). An explanation for the new type of critical behavior based on symmetry arguments is still lacking but numerical simulations suggest (Ódor, 2000; Hinrichsen, 2001c) the the behavior of this system can be described (at least for strong diffusion) by coupled subsystems: single particles performing annihilating random walks coupled to pairs (B) following the DP process: $B\to 2B$, $B\to\varnothing$. The model has two non-symmetric absorbing states: one is completely empty, while in the other a single particle walks randomly. Owing to this fluctuating absorbing state, this model does not oppose the conditions of the DP hypothesis. It was conjectured by Henkel and Hinrichsen (2001) that this kind of phase transition appears in models where (i) solitary particles diffuse, (ii) particle creation requires two particles, and (iii) particle removal requires at least two particles to meet. Other conditions that affect these classes are still under investigation.

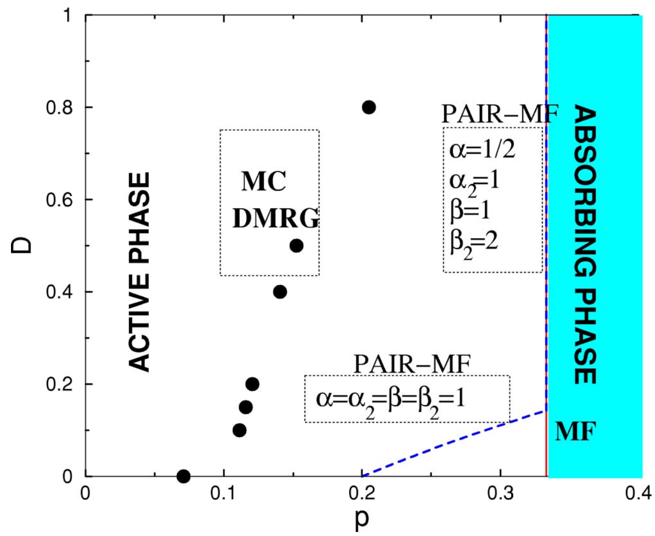

FIG. 22. (Color in online edition) Schematic phase diagram of the one-dimensional pair-contact process with particle diffusion model: circles, simulation and DMRG results; vertical solid line at $p=\frac{1}{3}$, mean-field; dashed line, pair approximation.

1. The PCPD model

A model similar to Eq. (206) was introduced in an early work by Grassberger (1982c). His preliminary simulations in one dimension showed a non-DP-type transition, but the model has been forgotten for a long time. The pair-contact particle-diffusion (PCPD) model introduced by Carlon *et al.* (2001) is controlled by two parameters, namely, the probability of pair annihilation p and the probability of particle diffusion D. The dynamical rules are

$$AA\varnothing, \quad \varnothing AA \to AAA \text{ with rate } (1-p)(1-D)/2,$$

$$AA \to \varnothing\varnothing \text{ with rate } p(1-D),$$

$$A\varnothing \leftrightarrow \varnothing A \text{ with rate } D. \tag{211}$$

The mean-field approximation gives a continuous transition at $p=1/3$. For $p\leqslant p_c(D)$ the particle and pair densities exhibit singular behavior:

$$\rho(\infty) \propto (p_c - p)^{\beta}, \quad \rho_2(\infty) \propto (p_c - p)^{\beta_2}, \tag{212}$$

while at $p=p_c(D)$ they decay as

$$\rho(t) \propto t^{-\alpha}, \quad \rho_2(t) \propto t^{-\alpha_2}, \tag{213}$$

with the exponents

$$\alpha = 1/2, \quad \alpha_2 = 1, \quad \beta = 1, \quad \beta_2 = 2. \tag{214}$$

According to pair mean-field approximations the phase diagram can be separated into two regions (see Fig. 22). While for $D>1/7$ the pair approximation gives the same $p_c(D)$ and exponents as the simple mean field, for

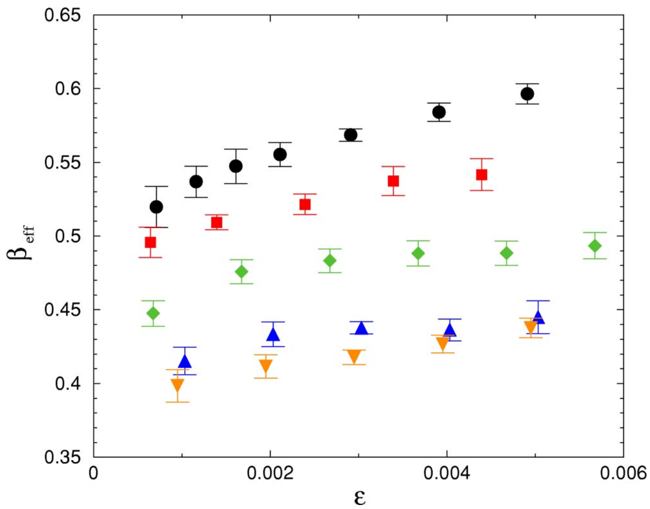

FIG. 23. (Color in online edition) Effective β exponents for different diffusion rates: ●, $D=0.05$; ■, $D=0.1$; ♦, $D=0.2$; ▲, $D=0.5$; ▼, $D=0.7$.

$D<1/7$ the transition line breaks and the exponents are different:

$$\alpha=1, \quad \alpha_2=1, \quad \beta=1, \quad \beta_2=1. \tag{215}$$

In the entire inactive phase the decay is characterized by the exponents

$$\alpha=1, \quad \alpha_2=2. \tag{216}$$

The DMRG (Carlon *et al.*, 2001) method and simulations of the one-dimensional PCPD model (Hinrichsen, 2001b) obtained $p_c(D)$ values in agreement, but for the critical exponents no clear picture was found. Carlon and Hinrichsen could not clarify whether the two distinct universality classes suggested by the pair mean-field approximations were really observable in the one-dimensional PCPD model. It is still a debated topic whether one new class, two new classes, or continuously changing exponents occur in one dimension. Since the model has two absorbing states (besides the vacuum state there is another with a single wandering particle), and some exponents were found to be close to those of the PC class ($Z=1.6-1.87, \beta/\nu_\perp=0.47-0.51$), Carlon *et al.* (2001) suspected that the transition (at least for low-D values) is PC type. However, the lack of Z_2 symmetry, parity conservation, and further numerical data (Ódor, 2000; Hinrichsen, 2001b) exclude this possibility. Note that the mean-field exponents are also different from those of the parity-conserving class. Simulations and coherent anomaly calculations for the one-dimensional $n=1$ annihilation-fission model (Ódor, 2000, 2003b) corroborated the two new universality class candidates (see Fig. 23 and Table XXI). The order-parameter exponent (β) seems to be very far from both the DP and the parity-conserving class values (Ódor, 2000, 2003b).

The two distinct class behaviors may be explained on the basis of competing diffusion strengths of particles and pairs (i.e., for large D's the explicit diffusion of lone particles is stronger). Similar behavior was observed in the case of one-dimensional models with coupled (conserved) diffusive fields (see Sec. V.H). However, a full agreement has not been achieved in the literature with respect to the precise values of the critical exponents. The low-D α is supported by Park and Kim (2002), who considered a case with coagulation and annihilation rates three times the diffusion rate. On the other hand, the high-D α of Table XXI coincides with that of Kockelkoren and Chate (2003a), who claim a single value for $0<D<1$. By assuming logarithmic corrections it was shown (Ódor, 2003b) that a single universality class can indeed be supported with the exponents

$$\alpha=0.21(1), \quad \beta=0.40(1),$$

$$Z=1.75(15), \quad \beta/\nu_\perp=0.38(1), \tag{217}$$

but there is no strong evidence for such corrections. Although the upper critical dimension is expected to be at $d_c=2$ (Ódor *et al.*, 2002), we cannot exclude the possibility of a second critical dimension ($d_c'=1$) or topological effects in one dimension that may cause logarithmic corrections to scaling. The spreading exponent η seems to change continuously with varying D. Whether this is true asymptotically or the effect of some huge correction to scaling is still not clear. The simulations of Ódor (2000) confirmed that it is irrelevant whether the particle production is spatially symmetric, $A\emptyset A\to AAA$, or spatially asymmetric, $AA\emptyset\to AAA$, $\emptyset AA\to AAA$. Recent simulations and higher-level generalized mean-field approximations suggest (Ódor *et al.*, 2002; Ódor, 2003b) that the peculiarities of the pair approximation are not real; for $N>2$ cluster approximations, the different scaling behavior in the low-D region disappears. Recently

TABLE XXI. Summary of results for one dimension, $n=1$ annihilation-fission model. The nonuniversal critical parameter p_c of the parallel model is shown here. CAM, coherent anomaly method.

D	0.05	0.1	0.2	0.5	0.9
p_c	0.25078	0.24889	0.24802	0.27955	0.4324
β_{CAM}		0.58(6)	0.58(2)	0.42(4)	
β	0.57(2)	0.58(1)	0.58(1)	0.40(2)	0.39(2)
δ	0.273(2)	0.275(4)	0.268(2)	0.21(1)	0.20(1)
η	0.10(2)		0.14(1)	0.23(2)	0.48(1)
δ'	0.004(6)		0.004(6)	0.008(9)	0.01(1)

two studies (Dickman and de Menezes, 2002; Hinrichsen, 2003) reported nonuniversality in the dynamical behavior of the PCPD model. While Dickman and de Menezes explored different sectors (a reactive and a diffusive one) in the time evolution and gave nontrivial exponent estimates, Hinrichsen offered a hypothesis that the ultimate long-time behavior should be characterized by DP behavior.

If we replace the annihilation process $2A \to \emptyset$ by coagulation $2A \to A$ in Eq. (206), we get the annihilation-coagulation model. Generalized mean-field approximations and simulations of this model resulted in similar phase diagram to those of the PCPD model, albeit without any sign of two distinct regions. In agreement with this, coherent anomaly approximations and simulations for the one-dimensional model found the same kind of continuous transition independently from D, with exponents in agreement with those of the PCPD in the low-D region (Ódor, 2001a; Park and Kim, 2002). Again the spatial symmetry of particle production was found to be irrelevant. An exact solution was found by Henkel and Hinrichsen (2001) for the special case in one dimension when the diffusion rate is equal to the coagulation rate, corresponding to the inactive phase according to which particle decay is like of ARW: $\rho \propto t^{-1/2}$.

2. Cyclically coupled spreading with pair annihilation

In this section I show an explicit two-component realization of the PCPD class. A cyclically coupled two-component reaction-diffusion system was introduced by Hinrichsen (2001c),

$$A \to 2A, \quad A \to \emptyset, \quad A \to B,$$
$$2B \to A, \quad B\emptyset \leftrightarrow \emptyset B, \tag{218}$$

which mimics the PCPD model (Sec. V.F.1) by mapping pairs to A's and single particles to B's. This model is a coupled DP+ARW system. Its (1+1)-dimensional critical space-time evolution pattern looks very similar to that of the PCPD model. The appearance of evolution in space-time seems to be a particular feature of this class. It is built up from compact domains with a cloud of lone particles wandering and interacting with them. Furthermore, this model also has two nonsymmetric absorbing states: a completely empty one and another with a single wandering B. When the annihilation and diffusion rates of B's ($r=D=1$) are fixed, the model exhibits a continuous phase transition by varying the production rate of A's and the $A \to B$ transmutation rate. The simulations in one dimension showed that $\rho_A \propto \rho_B$ for large times and resulted in the following critical exponent estimates:

$$\alpha = 0.21(2), \quad \beta = 0.38(6), \quad \beta' = 0.27(3),$$
$$Z = 1.75(5), \quad \nu_{\|} = 1.8(1), \tag{219}$$

satisfying the generalized hyperscaling relation (155). These exponents are similar to those of the PCPD model in the high-diffusion region (see Table XXI), which is reasonable since $D=1$ is fixed here.

3. The parity-conserving annihilation-fission model

As we have seen, parity conservation plays an important role in unary production systems. In the case of BARW processes it changes the universality of the transition from directed percolation (Sec. IV.A.3) to the parity-conserving class (Sec. IV.D.1). The question arises whether one can see similar behavior in the case of binary production systems. Recently Park *et al.* (2001) investigated a parity-conserving representative ($n=2$) of the one-dimensional annihilation-fission model [Eq. (206)]. By performing simulations for low D's they found critical exponents in the range of values determined for the corresponding PCPD class. Park *et al.* claim that the conservation law does not affect the critical behavior and that the binary nature of the offspring production is a sufficient condition for this class (see, however, Sec. V.F.2, where there is no such condition).

The two-dimensional version of the parity-conserving annihilation-fission model was investigated by generalized mean-field and simulation techniques (Ódor *et al.*, 2002). While the $N=1,2$ mean-field approximations showed similar behavior to that of the PCPD model (Sec. V.F.1), including the two-class prediction for $N=2$, the $N=3,4$ approximations do not show D dependence of the critical behavior: $\beta=1, \beta_2=2$ were obtained for $D>0$. Large-scale simulations of the particle density confirmed the mean-field scaling behavior with logarithmic corrections. This result can be interpreted as numerical evidence supporting an upper critical dimension in this model of $d_c=2$. The pair density decays in a similar way, but with an additional logarithmic factor to the order parameter. This kind of strongly coupled behavior at and above criticality was observed in case of the pair-contact model too (see Sec. V.E). At the $D=0$ end point of the transition line (2+1)-dimensional-class DP criticality (see Sec. IV.A) was found for ρ_2 and for $\rho-\rho(p_c)$. In the inactive phase for $\rho(t)$ we can observe the two-dimensional ARW class scaling behavior (see Sec. IV.C.1), while the pair density decays as $\rho_2 \propto t^{-2}$. Again, as in $d=1$, parity conservation seems to be irrelevant.

G. BARWe with coupled nondiffusive field class

Similarly to the pair-contact process (Sec. V.E), the effect of infinitely many frozen absorbing states has been investigated in the case of a BARWe model. A parity-conserving version of the one-dimensional pair-contact model (Sec. V.E) was introduced by Marques and Mendes (1999), in which pairs follow a BARW2 process, while lone particles are frozen. Simulations showed that while the critical behavior of pairs in the case of homogeneous, random initial distribution belongs to the parity-conserving class (Sec. IV.D.1), the spreading exponents satisfy hyperscaling [Eq. (155)] and change continuously when the initial particle density is varied. These results are similar to those found in the pair-contact model. Again long-memory effects are responsible for the nonuniversal behavior in case of seedlike initial conditions. The slowly decaying memory was con-

TABLE XXII. Summary of results for diffusive conserved-field classes for $D_A=D_B$.

d	β	Z	$\nu_\perp$
1	0.44(1)	2	2.1(1)
$4-\epsilon$	$1-\epsilon/8$	2	$2/d$

TABLE XXIII. Summary of results for diffusive conserved-field classes for $D_A<D_B$.

d	β	Z	$\nu_\perp$
1	0.33(2)		2
$4-\epsilon$	1	2	$2/d$

firmed by studying a one-dimensional, interacting monomer-monomer model (Park and Park, 2001) by simulations.

H. Directed percolation with diffusive, conserved slave field classes

In view of the results of model C (Sec. III), the obvious question is, what happens to the phase transition to an absorbing state of a reaction-diffusion system if a conserved secondary density is coupled to a nonconserved order parameter? One can deduce from the BARW1 spreading process (Sec. IV.A.3) a two-component, reaction-diffusion model, the diffusive conserved-field model (Kree *et al.*, 1989; Wijland *et al.*, 1998), that exhibits total particle density conservation as follows:

$$A+B\xrightarrow{k}2B,\quad B\xrightarrow{1/\tau}A. \tag{220}$$

When the initial particle density $[\rho=\rho_A(0)+\rho_B(0)]$ is varied, a continuous phase transition occurs. The general field-theoretical case was investigated by Wijland *et al.* (1998), while the equal-diffusion case, $D_A=D_B$, was studied by Kree *et al.* (1989). The mean-field exponents that are valid above $d_c=4$ are shown in Table XXII. The rescaled action of this model is

$$\begin{aligned} S[\varphi,\bar\varphi,\psi,\bar\psi]=\int d^dx dt\{&\bar\varphi(\partial_t-\Delta)\varphi+\bar\psi[\partial_t+\lambda(\varphi-\Delta)]\psi\\ &+\mu\bar\varphi\Delta\psi+g\psi\bar\psi(\psi-\bar\psi)+u\psi\bar\psi(\varphi+\bar\varphi)\\ &+v_1(\psi\bar\psi)^2+v_2\psi\bar\psi(\psi\bar\varphi-\bar\psi\varphi)+v_3\varphi\bar\varphi\psi\bar\psi\\ &-\rho_B(0)\delta(t)\bar\psi\}, \end{aligned} \tag{221}$$

where ψ and ϕ are auxiliary fields, defined such that their average values coincide with the average density of B particles and the total density of particles, respectively. The coupling constants are related to the original parameters of the master equation by

$$\begin{aligned} &\mu=1-D_B/D_A,\quad g=k\sqrt{\rho}/D_A,\quad \lambda\sigma=k(\rho_c^{\rm mf}-\rho)/D_A,\\ &v_1=v_2=-v_3=k/D_A,\quad u=-k\sqrt{\rho}/D_A,\quad \lambda=D_B/D_A,\\ &\rho_B(0)=\rho_B(0)/\sqrt{\rho}. \end{aligned} \tag{222}$$

If one omits from the action [Eq. (221)] the initial time term proportional to $\rho_B(0)$, then the remainder is, for $\mu=0$ (i.e., $D_A=D_B$), invariant under time-reversal symmetry:

$$\begin{aligned} \psi(x,t)&\to-\bar\psi(x,-t),\\ \bar\psi(x,t)&\to-\psi(x,-t),\\ \varphi(x,t)&\to\bar\varphi(x,-t),\\ \bar\varphi(x,t)&\to\varphi(x,-t). \end{aligned} \tag{223}$$

The epsilon expansion solution (Kree *et al.*, 1989) and simulation results (de Freitas *et al.*, 2000; Fulco *et al.*, 2001) are summarized in Table XXII. Interestingly the RG predicts $Z=2$ and $\nu_\perp=2/d$ in all orders of perturbation theory.

The breaking of this symmetry for $\mu\neq0$, that is, when the diffusion constants D_A and D_B are different, causes different critical behavior for this system. For $D_A<D_B$, the renormalization group (Wijland *et al.*, 1998) predicts new classes with $Z=2$, $\beta=1$, $\nu_\perp=2/d$, but simulations in one dimension (Fulco *et al.*, 2001) show different behavior (see Table XXIII). For non-Poissonian initial particle density distributions, the critical initial slip exponent η varies continuously with the width of the distribution of the conserved density. The $D_A=0$ extreme case is discussed in Sec. V.I.

For $D_A>D_B$ no stable fixed-point solution was found by renormalization, hence Wijland *et al.* (1998) conjectured that there was a first-order transition, for which signatures were found in two dimensions by simulations (Oerding *et al.*, 2000). However, ϵ expansion may break down in the case of another critical dimension $d_c'<d_c=4$, for which simulations in one dimension (Fulco *et al.*, 2001) have provided numerical support (see Table XXIV).

I. Directed percolation with frozen, conserved slave field classes

If the conserved field coupled to the BARW1 process [Eq. (220)] is nondiffusive, non-DP-class behavior is

TABLE XXIV. Summary of results for diffusive conserved-field classes for $D_A>D_B$.

d	β	Z	$\nu_\perp$
1	0.67(1)		2
$4-\epsilon$	0		

TABLE XXV. Summary of results for nondiffusive conserved-field classes.

d	α	β	γ	Z	$\nu_{\|}$	ς	η
1	0.14(1)	0.28(1)		1.5(1)	2.5		
2	0.50(5)	0.64(1)	1.59(3)	1.55(4)	1.29(8)	2.22(3)	0.29(5)
3	0.90(3)	0.84(2)	1.23(4)	1.75(5)	1.12(8)	2.0(4)	0.16(5)
4	1	1	1	2	1	2	0

again reported (Pastor-Satorras and Vespagnani, 2000; Rossi *et al.*, 2000). This is known as the nondiffusive conserved-field (NDCF) class. The corresponding action can be derived from Eq. (223) in the $D_A=0$ limit:

$$S=\int d^d x dt[\bar{\varphi}(\partial_t + r - D\nabla^2)\varphi + \bar{\psi}(\partial_t - \lambda\nabla^2)\psi + g\psi\bar{\psi}(\psi - \bar{\psi}) + u\psi\bar{\psi}(\varphi+\bar{\varphi}) + v_1(\psi\bar{\psi})^2 + v_2\psi\bar{\psi}(\psi\bar{\varphi} - \bar{\psi}\varphi) + v_3\varphi\bar{\varphi}\psi\bar{\psi}]. \tag{224}$$

When we neglect irrelevant terms, Eq. (224) is invariant under the shift transformation,

$$\psi \to \psi + \Delta, \quad r \to r - v_2\Delta, \tag{225}$$

where Δ is any constant. The field-theoretical analysis of this action has run into difficulties (Pastor-Satorras and Vespagnani, 2000). The main examples of models in the NDCF class are the conserved threshold transfer process and the conserved reaction-diffusion model (Pastor-Satorras and Vespagnani, 2000; Rossi *et al.*, 2000). Furthermore, the models described by the NDCF class embrace a large group of stochastic sandpile models (Jensen, 1998), in particular, fixed-energy Manna models (Manna, 1991; Dickman *et al.*, 1998; Dhar, 1999; Muñoz *et al.*, 2001). The upper critical dimension $d_c=4$ has been confirmed by simulations (Lübeck and Hucht, 2002).

It was also shown (Alava and Munoz, 2001) that these models describe the depinning transition of the quenched Edwards-Wilkinson class (see Sec. VI.C) or linear interface models (Barabási and Stanley, 1995; Halpin-Healy and Zhang, 1995) owing to the fact that quenched disorder can be mapped onto long-range temporal correlations in the activity field (Marsili, 1994). However, this mapping could not be done on the level of Langevin equations of the representatives of NDCF and linear interface models, and in one dimension this equivalence may break down (Alava and Munoz, 2001; Dickman, Alava, *et al.*, 2001; Kockelkoren and Chaté, 2003b). The critical exponents determined by simulations (Pastor-Satorras and Vespagnani, 2000; Rossi *et al.*, 2000; Lübeck, 2001, 2002; Dickman, Tomé, and Oliveira, 2002) and mean-field plus coherent anomaly methods in one dimension (Dickman, 2002) are summarized in Table XXV. Similarly to the pair-contact process, these models exhibit infinitely many absorbing states. Therefore nonuniversal spreading exponents are expected (in Table XXV the exponent η corresponding to natural initial conditions is shown).

J. Coupled *N*-component DP classes

From the basic reaction-diffusion systems one can generate N-component ones coupled by interactions symmetrically or asymmetrically. Janssen (1997b, 2001) introduced and analyzed by the field-theoretical RG method (up to two-loop order) a quadratically coupled, N-species generalization of the DP process of the form

$$A_\alpha \leftrightarrow 2A_\alpha,$$
$$A_\alpha \to \emptyset,$$
$$A_\alpha + A_\beta \to kA_\alpha + lA_\beta, \tag{226}$$

where k,l may take the values (0,1). He has shown that the multicritical behavior is always described by Reggeon field theory (DP class), but that this is unstable and leads to unidirectionally coupled DP systems (see Fig. 24).

He has also shown that by this model linearly, unidirectionally coupled directed percolation (see Sec. VI.H) can be described. The universality class behavior of unidirectionally coupled DP is discussed in Sec. VI.H.

In one dimension, if BARWo-type processes are coupled (which alone exhibit DP class transitions (see Sec. IV.A.3), hard-core interactions can modify the phase-transition universality (see Sec. V.L.1).

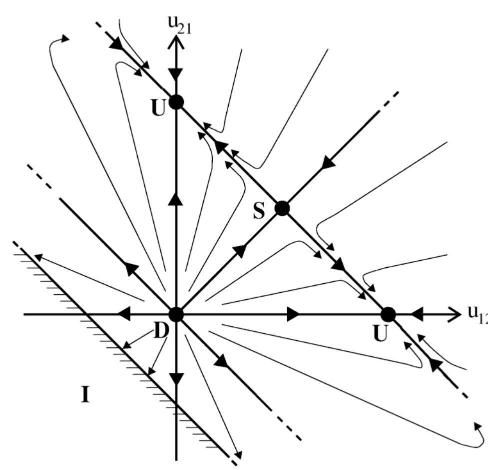

FIG. 24. (Color in online edition) Flow of the interspecies couplings in the two-component, DP model under renormalization: D, decoupled; S, symmetric; U, unidirectional fixed points. From Janssen, 2001.

K. Coupled N-component BARW2 classes

Bosonic, N-component BARW systems with two offspring (N-BARW2), of the form

$$A_\alpha \to 3A_\alpha, \tag{227}$$

$$A_\alpha \to A_\alpha 2A_\beta, \tag{228}$$

$$2A_\alpha \to \varnothing \tag{229}$$

were introduced and investigated by Cardy and Täuber (1998) via the field-theoretical RG method. These models exhibit parity conservation of each species and permutation symmetry on N types [the generalization for O(N) symmetry is violated by the annihilation term]. The $A \to 3A$ process turns out to be irrelevant, because like pairs annihilate immediately. Models with Eq. (228) branching terms exhibit continuous phase transitions at zero branching rate. The universality class is expected to be independent from N and coincides with that of the $N \to \infty$ (N-BARW2) model, which can be solved exactly. The critical dimension is $d_c=2$, and for $d \leqslant 2$ the exponents are

$$\beta = 1, \quad Z = 2, \quad \alpha = d/2, \quad \nu_\| = 2/d, \quad \nu_\perp = 1/d, \tag{230}$$

while at $d=d_c=2$ logarithmic corrections to the density decay are expected. Simulations on a $d=2$ (fermionic) lattice model confirmed these results (Ódor, 2001c).

In one dimension it turns out that introducing hard-core interactions between particles, so that two particles can never overlap, can be relevant and different universal behavior can emerge. Here the phase transition behavior of the fermionic and bosonic models are equivalent in one dimension (at least for static exponents) in the case of pairwise initial conditions (see Sec. V.B) when different types of particles do not make up blockades for each other. Such a situation happens when these particles are generated as domain walls of $(N+1)$-component systems exhibiting S_{N+1} symmetric absorbing states (see Secs. V.K and IV.D.2).

The generalized contact process has already been introduced in Sec. IV.D.3 with the main purpose of showing an example of a parity-conserving universality class transition in the case of Z_2-symmetric absorbing states. The more general case with $n>2$ permutation symmetric absorbing states was investigated, using the DMRG method, by Hooyberghs *et al.* (2001) and turned out to exhibit an N-BARW2 transition. In the one-dimensional model, where each lattice site can be occupied by at most one particle (A) or can be in any of n inactive states ($\varnothing_1, \varnothing_2, \dots \varnothing_n$), the reactions are

$$AA \to A\varnothing_k, \varnothing_k A \ \text{ with rate } \lambda/\mathrm{n}, \tag{231}$$

$$A\varnothing_k, \varnothing_k A \to \varnothing_k\varnothing_k \ \text{ with rate } \mu_k, \tag{232}$$

$$A\varnothing_k, \varnothing_k A \to AA \ \text{ with rate } 1, \tag{233}$$

$$\varnothing_k\varnothing_l \to A\varnothing_l, \quad \varnothing_k A \ \ (k \neq l) \text{ with rate } 1. \tag{234}$$

The original contact process (Sec. IV.A.1) corresponds to the $n=1$ case, from which the reaction (234) is obviously absent. The reaction (234) in the case $n \geqslant 2$ ensures that configurations like $(\varnothing_i\varnothing_i \cdots \varnothing_i\varnothing_i\varnothing_j\varnothing_j \cdots \varnothing_j\varnothing_j)$, with $i \neq j$, are not absorbing. Such configurations do evolve in time until the different domains coarsen and one of the n absorbing states $(\varnothing 1 \varnothing 1 \cdots \varnothing 1)$, $(\varnothing 2 \varnothing 2 \cdots \varnothing 2)$, $(\varnothing n \varnothing n \cdots \varnothing n)$ is reached. For generalized contact processes with $n=2$, simulations (Hinrichsen, 1997) and a DMRG study (Hooyberghs *et al.*, 2001) proved that the transition falls in the parity-conserving class if $\mu_1 = \mu_2$, or in the DP class if the symmetry between the two absorbing states is broken ($\mu_1 \neq \mu_2$).

The DMRG study for $n=3$ and $n=4$ showed that the model is in the active phase over the whole parameter space, and the critical point is shifted to the limit of infinite reaction rates. In this limit the dynamics of the model can be mapped onto the zero-temperature n-state Potts model (see also the simulation results of Lipowski and Droz, 2002a). It was conjectured by Hooyberghs *et al.* (2001) that the model is in the same N-BARW2 universality class for all $n \geqslant 3$. If we consider the region between $\varnothing_i$ and $\varnothing_j$ as a domain wall (X_{ij}) one can follow the dynamics of such variables. In the limit $\lambda \to \infty$, X_{ij} coincides with the particle A, and in the limit $\mu \to \infty$, X_{ij} coincides with the bond variable $\varnothing_i\varnothing_j$. For finite values of these parameters one can still apply this reasoning at a coarse-grained level. In this case X_{ij} is not a sharp domain wall, but an object with a fluctuating thickness. For $n=2$ it was shown in Sec. IV.D.3 that the variables of such a domain wall follow BARW2 dynamics [Eq. (175)]. For $n>2$ one can show that besides the N-BARW2 reactions (227) and (228) involving a maximum of two types of particles, reaction types occur involving three different domains ($i \neq j$, $i \neq k$ and $j \neq k$):

$$X_{ij} \to X_{ik}X_{kj}, \quad X_{ik}X_{kj} \to X_{ij} \tag{235}$$

with increasing importance as $n \to \infty$. These reactions break the parity conservation of the N-BARW2 process. Therefore the numerical findings of Hooyberghs *et al.* (2001) for $n=3,4$ indicate that they are probably irrelevant or the conditions for N-BARW2 universal behavior could be relaxed. Owing to the fact that the X_{ij} variables are domain walls, they appear in a pairwise manner, hence hard-core exclusion effects are ineffective for the critical behavior in one dimension. For the effect of pairwise initial conditions for dynamical exponents, see Sec. V.B.

For $n=3$, upon breaking the global S_3 symmetry to a lower one, one gets a transition either in the directed percolation (Sec. IV.A) or in the parity-conserving class (Sec. IV.D), depending on the choice of parameters (Hooyberghs *et al.*, 2001). Simulations indicate (Lipowski and Droz, 2002b) that for this model local symmetry breaking may also generate a parity-conserving-class transition).

TABLE XXVI. Summary of critical exponents in one dimension for N-component branching and annihilating random walk with even offspring (N-BARW2)-like models. The N-BARW2 data are quoted from Cardy and Täuber (1996). Data divided by "|" correspond to random vs pairwise initial conditions (Ódor and Menyhárd, 2000; Hooyberghs *et al.*, 2001; Menyhárd and Ódor, 2002). Exponents denoted by * exhibit initial-density dependence.

Exponent	N-BARW2	N-BARW2s	N-BARW2a
$\nu_{\parallel}$	2	2.0(1)|0.915(2)	8.0(4)|3.66(2)
Z	2	4.0(2)|1.82(2)*	4.0(2)|1.82(2)*
α	1/2	0.25(1)|0.55(1)*	0.25(1)|0.55(1)*
β	1	0.50(1)|1.00(1)	2.0(1)|1.00(1)

L. Hard-core 2-BARW2 classes in one dimension

Besides the effects of coupling interactions in low dimensions, blockades generated by hard-core particles may also play an important role. The effect of particle exclusion (i.e., $AB\leftrightarrow BA$) in 2-BARW2 models (Sec. V.K) was investigated by Kwon *et al.* (2000) and Ódor (2001c). For $d=2$ the bosonic field-theoretical predictions (Cardy and Täuber, 1998) were confirmed (Ódor, 2001c; mean-field class transition with logarithmic corrections). In one dimension, however, two types of phase transitions were identified at zero branching rate ($\sigma=0$) depending on the arrangement of offspring relative to the parent in process (228). That is, if the parent separates the two offspring (2-BARW2s),

$$A \overset{\sigma}{\rightarrow} BAB, \tag{236}$$

the steady-state density is higher than in the case when they are created on the same site (2-BARW2a):

$$A \overset{\sigma}{\rightarrow} ABB, \tag{237}$$

at a given branching rate, because in the former case they are unable to annihilate each other. This results in different order-parameter exponents for the symmetric (2-BARW2s) and asymmetric (2-BARW2a) cases,

$$\beta_s = 1/2, \quad \beta_a = 2. \tag{238}$$

This result is in contrast with a widespread belief that the bosonic field theory (where $AB\leftrightarrow BA$ is allowed) can describe these systems (because in that case the critical behavior is different [Eq. (230)]). This observation led Kwon *et al.* (2000) to the conjecture that in one-dimensional, reaction-diffusion systems a series of new universality classes should appear if particle exclusion is present. Note, however, that since the transition is at $\sigma=0$ in both cases, the on-critical exponents do not depend on how particles are created and they can be identified with those described in Sec. V.B. Ódor (2001c) determined a set of critical exponents satisfying scaling relations for these two new classes shown in Table XXVI.

1. Hard-core 2-BARWo models in one dimension

Hard-core interactions in the two-component, one-offspring production model (2-BARW1) were investigated by Ódor (2001b). Without interaction between different species one would expect a DP-class transition. By introducing the $AB\leftrightarrow BA$ blocking to the two-component model,

$$A \overset{\sigma}{\rightarrow} AA, \quad B \overset{\sigma}{\rightarrow} BB, \tag{239}$$

$$AA \overset{1-\sigma}{\rightarrow} \emptyset, \quad BB \overset{1-\sigma}{\rightarrow} \emptyset, \tag{240}$$

Ódor (2001b) located a DP-class transition at $\sigma=0.81107$. Note that the effects exerted by different species on each other are irrelevant now, unlike for the case of a coupled ARW (Sec. V.B). On the other hand, if we couple the two subsystems by production of offspring:

$$A \overset{\sigma/2}{\rightarrow} AB, \quad A \overset{\sigma/2}{\rightarrow} BA, \tag{241}$$

$$B \overset{\sigma/2}{\rightarrow} AB, \quad B \overset{\sigma/2}{\rightarrow} BA, \tag{242}$$

$$AA \overset{1-\sigma}{\rightarrow} \emptyset, \quad BB \overset{1-\sigma}{\rightarrow} \emptyset, \tag{243}$$

a continuous phase transition emerges when the parameter describing the offspring production rate σ is zero. Therefore the critical exponents are the same as those described in Sec. V.B—and the order-parameter exponent is found to be $\beta=1/2$. Therefore this transition belongs to the same class as the 2-BARW2s model (see Sec. V.L). The parity conservation law, which is relevant in the case of one-component BARW systems, turns out to be irrelevant here. This finding reduces the expectations suggested by Kwon *et al.* (2000) for a whole new series of universality classes in one-dimensional systems with exclusions. In fact, the blockades introduced by exclusions generate robust classes. Ódor (2001b) suggested that in coupled branching and annihilating random-walk systems of N types of excluding particles for continuous transitions at $\sigma=0$, two universality classes exist, those of 2-BARW2s and 2-BARW2a models, depending on whether the reactants can immediately annihilate (i.e., whether similar particles are separated by other types of particles) or not. Recent investigations in similar models (Lipowski and Droz, 2001; Park and Park, 2001) are in agreement with this hypothesis.

2. Coupled binary spreading processes

Two-component versions of the PCPD model (Sec. V.F.1) with particle exclusion in one dimension were introduced and investigated by simulations (Ódor, 2002) with the aim of testing whether the hypothesis of Ódor (2001b) for N-component BARW systems (Sec. V.L.1) can be applied to such models. The following models with the same diffusion and annihilation terms ($AA\rightarrow\emptyset$, $BB\rightarrow\emptyset$) as in Sec. V.B and different production processes were investigated.

(1) Production and annihilation random-walk model (2-PARW):

$$AA \xrightarrow{\sigma/2} AAB, \quad AA \xrightarrow{\sigma/2} BAA, \tag{244}$$

$$BB \xrightarrow{\sigma/2} BBA, \quad BB \xrightarrow{\sigma/2} ABB, \tag{245}$$

(2) Symmetric production and annihilation random-walk model (2-PARWs):

$$AA \xrightarrow{\sigma} AAA, \tag{246}$$

$$BB \xrightarrow{\sigma} BBB, \tag{247}$$

These two models exhibit active steady states for $\sigma>0$ with a continuous phase transition at $\sigma_c=0$. Therefore the exponents defined on the critical point are those of the two-component ARW model with exclusion (Sec. V.B). Together with the exponent $\beta=2$ result for both cases this indicates that they belong to the N-BARW2a class. This also means that the hypothesis set up for N-BARW2 systems (Ódor, 2001b; Sec. V.L.1) can be extended.

(3) Asymmetric production and annihilation random-walk model (2-PARWa)

$$AB \xrightarrow{\sigma/2} ABB, \quad AB \xrightarrow{\sigma/2} AAB, \tag{248}$$

$$BA \xrightarrow{\sigma/2} BAA, \quad BA \xrightarrow{\sigma/2} BBA. \tag{249}$$

This model does not have an active steady state. The AA and BB pairs annihilate themselves on contact, while if an A and B particle meet an $AB\to ABB \to A$ process eliminates blockades, and the densities decay with the $\rho \propto t^{-1/2}$ law for $\sigma>0$. For $\sigma=0$ the blockades persist, and in case of a random initial state $\rho \propto t^{-1/4}$ decay (see Sec. V.B) can be observed.

(4) Asymmetric production and annihilation random-walk model with spatially symmetric creation (2-PARWas):

$$AB \xrightarrow{\sigma/2} ABA, \quad AB \xrightarrow{\sigma/2} BAB, \tag{250}$$

$$BA \xrightarrow{\sigma/2} BAB, \quad BA \xrightarrow{\sigma/2} ABA. \tag{251}$$

In this case AB blockades proliferate from production events. As a consequence, an active steady state appears for $\sigma>0.3253(1)$ with a continuous phase transition. The space-time evolution from a random initial state shows (Fig. 25) that compact domains of alternating $..ABAB..$ sequences are formed, separated by lone wandering particles. This pattern is very similar to what was seen in the case of one-component binary spreading processes (Hinrichsen, 2001c): compact domains within a cloud of lone random walkers, except that now domains are built up from alternating sequences only. This means that the $..AAAA\ldots$ and $\ldots BBBB\ldots$ domains decay by this annihilation rate and particle blocking is responsible for the formation of compact clusters. In the language of the coupled DP+ARW model (Hinrichsen, 2001c), the pairs following the DP process are now the AB pairs, which cannot decay spontaneously but only through an annihilation process: $AB+BA \to \emptyset$. They interact with two types of particles executing annihilating random walks with exclusions. The simulations resulted in the critical exponent estimates: $\beta=0.37(2)$, $\alpha=0.19(1)$, and $Z=1.81(2)$, which agree fairly well with those of the PCPD model in the high-diffusion-rate region (Ódor, 2000).

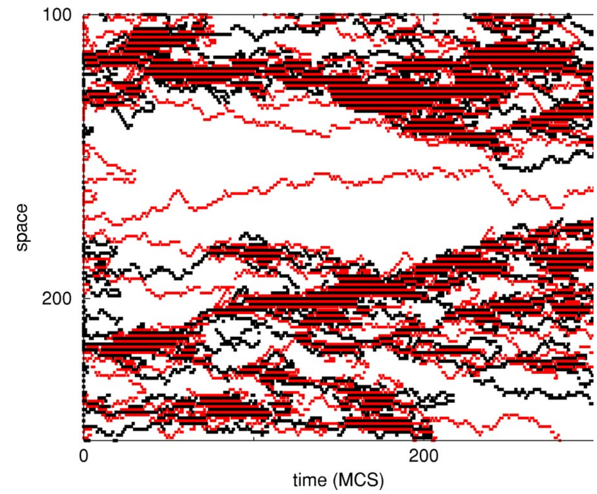

FIG. 25. (Color in online edition) Space-time evolution of the asymmetric production and m-particle annihilating random walk (2-PARWas) model at the critical point. Black pixels correspond to A particles, others to B's. From Ódor, 2002.

VI. INTERFACE GROWTH CLASSES

Interface growth classes are strongly related to the basic universality classes discussed so far and can be observed in experiments more easily. For example, one of the few experimental realizations of the robust DP class (Sec. IV.A) is related to a depinning transition in inhomogeneous porous media (Buldyrev *et al.*, 1992; see Sec. VI.F). The interface models can be defined either by continuum equations or by lattice models of solid-on-solid (SOS) or restricted solid-on-solid (RSOS) types. In the latter case the height variables h_i of adjacent sites are restricted,

$$|h_i - h_{i+1}| \leqslant 1. \tag{252}$$

The morphology of a growing interface is usually characterized by its width,

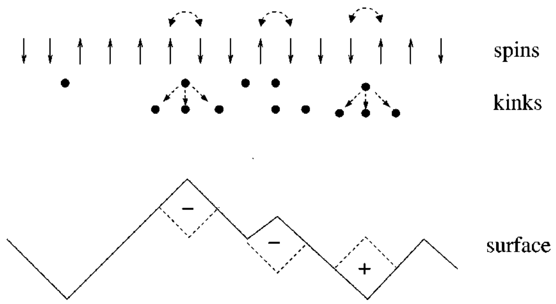

FIG. 26. Mapping between spins, kinks and surfaces.

$$W(L,t)=\left[\frac{1}{L}\sum_i h_i^2(t)-\left(\frac{1}{L}\sum_i h_i(t)\right)^2\right]^{1/2}. \qquad (253)$$

In the absence of any characteristic length, growth processes are expected to show power-law behavior of the correlation functions in space and height and the surface is described by the *Family-Vicsek scaling* (Family and Vicsek, 1985) form:

$$W(L,t)=t^{\tilde{\alpha}/Z}f[L/\xi_{\|}(t)], \qquad (254)$$

with the scaling function $f(u)$

$$f(u)\sim\begin{cases}u^{\tilde{\alpha}} & \text{if } u\ll 1\\ \text{const} & \text{if } u\gg 1.\end{cases} \qquad (255)$$

Here $\tilde{\alpha}$ is the roughness exponent and characterizes the stationary regime in which the correlation length $\xi_{\|}(t)$ has reached a value larger than the system size L. The ratio $\tilde{\beta}=\tilde{\alpha}/Z$ is called the *growth exponent* and characterizes the short-time behavior of the surface. Similarly to equilibrium critical phenomena, these exponents do not depend on the microscopic details of the system under investigation. Using these exponents it is possible to divide growth processes into universality classes (Barabási and Stanley, 1995; Krug, 1997). The scaling form (254) of W^2 is invariant under Λ rescaling,

$$x\to\Lambda x,\quad t\to\Lambda^Z t,\quad h(x,t)\to\Lambda^{-\tilde{\alpha}}h(x,t). \qquad (256)$$

Recently *anomalous roughening* has been observed in many growth models and experiments. In these cases the measurable $\tilde{\alpha}_{loc}$ roughness exponent is different from $\tilde{\alpha}$ and may satisfy a different scaling law.[2]

Surfaces in $(d+1)$-dimensional systems can be mapped onto a time step of a d-dimensional particle reaction-diffusion or spin models. For example, the one-dimensional Kawasaki spin model corresponding to the $K\leftrightarrow 3K$ process with random walk of kinks is mapped onto the one-dimensional surface as shown in Fig. 26.

[2]See, for example, Oliveira, 1992; Schroeder *et al.*, 1993; Yang, Wang, and Lu, 1994; Dasgupta *et al.*, 1996; Jeffries *et al.*, 1996; López and Rodríguez, 1996, 1997; Sarma *et al.*, 1996; Krug, 1997; Bru *et al.*, 1998; López and Schmittbuhl, 1998; Morel *et al.*, 1998.

This means that a spatial profile $\{h_i(t)\}$ corresponds to a unique $\{s_j\}$ spin configuration at time t by accumulating the spin values

$$h_i(t)=\sum_{j=1}^{i}s_j. \qquad (257)$$

In one dimension the surface can also be considered as a random walker with fluctuation

$$\Delta x\propto t^{1/Z_w}. \qquad (258)$$

Hence the roughness exponent is related to the dynamical exponent Z_w as

$$\tilde{\alpha}=1/Z_w. \qquad (259)$$

The $\tilde{\alpha}=1/2$ corresponds to uncorrelated (or finite correlation length) random walks. If $\tilde{\alpha}>1/2$ the surface exhibits correlations, while if $\tilde{\alpha}<1/2$ the displacements in the profile are anticorrelated. Since the surfaces may exhibit drifts, fluctuations around the mean are measured defining the *local roughness (Hurst) exponent*. Using this surface mapping, Sales *et al.* (1997) have characterized the different classes of Wolfram's one-dimensional cellular automata (Wolfram, 1983).

One can show that by coarse graining the one-dimensional Kawasaki dynamics,

$$w_i=\frac{1}{4\tau}[1-\sigma_i\sigma_{i+1}+\lambda(\sigma_{i+1}-\sigma_i)], \qquad (260)$$

a mapping can be made onto the Kardar-Parisi-Zhang equation (270), and the surface dynamics for $\lambda\neq 0$ (corresponding to the anisotropic case) are in the Kardar-Parisi-Zhang class (Sec. VI.D), while for $\lambda=0$ they are in the Edwards-Wilkinson class (Sec. VI.B). While these classes are related to the simple random walk with $Z_w=1/\tilde{\alpha}=2$, the question arises what surfaces are related to other kinds of random walks (for example, Levy flights or correlated random walks, etc.). Recently it was shown that globally constrained random walks (i.e., in which a walker needs to visit each site an even number of times) can be mapped onto surfaces with dimer-type dynamics (Noh *et al.*, 2001) with $Z_w=3=1/\tilde{\alpha}$.

By studying the correspondences between lattice models with absorbing states and models of pinned interfaces in random media, Dickman and Munoz (2000) established the scaling relation

$$\tilde{\beta}=1-\beta/\nu_{\|}, \qquad (261)$$

which was confirmed numerically for $d=1,2,3,4$ contact processes (Sec. IV.A.1). The local roughness exponent was found to be smaller than the global value, indicating anomalous surface growth in DP-class models.

In interface models different types of transitions may take place:

(1) *Roughening transitions* may occur between the smooth phase characterized by finite width W (in an infinite system) and the rough phase when the width diverges in an infinite system (but saturates in finite ones) as a result of varying some control parameters (ϵ). Near

the transition point the spatial ($\xi_{\parallel}$) and growth direction correlations ($\xi_{\perp}$) diverge as

$$\xi_{\parallel} \propto \epsilon^{\nu_{\parallel}}, \tag{262}$$

$$\xi_{\perp} \propto \epsilon^{\nu_{\perp}} \tag{263}$$

(note that in RD systems $\xi_{\parallel}$ denotes temporal correlation length). In the smooth phase the heights $h_i(t)$ are correlated below $\xi_{\perp}$. While in equilibrium models roughening transitions exist in $d>1$ dimensions only in nonequilibrium models this may occur in $d=1$ as well.

(2) *Depinning transitions* occur when, as the consequence of changing some control parameter (usually an external force F), the surface starts propagating with speed v and evolves into a rough state. Close to the transition v is expected to scale as

$$v \propto (F-F_c)^{\tilde{\theta}} \tag{264}$$

with the $\tilde{\theta}$ velocity exponent and the correlation exponents diverge. Known depinning transitions (in random media) are related to absorbing phase transitions with conserved quantities (see Secs. VI.C and VI.E).

(3) A so-called *faceting phase transition* may also take place in the rough phase when up-down symmetrical facets appear. In this case the surface scaling behavior changes (see Sec. VI.I).

A. The random deposition class

Random deposition is the simplest surface growth process that involves uncorrelated adsorption of particles on top of each other. Therefore columns grow independently, linearly without bounds. The roughness exponent $\tilde{\alpha}$ (and correspondingly Z) is not defined here. The width of the surface grows as $W \propto t^{1/2}$ hence $\tilde{\beta}=1/2$ in all dimensions. An example of such behavior is the dimer growth model described in Sec. VI.I.

B. Edwards-Wilkinson classes

As was mentioned in Sec. VI, growth models of this class can easily be mapped onto spins with symmetric Kawasaki dynamics or onto particles with annihilating random walks (Sec. IV.C.1). If we postulate the translation and reflection symmetries

$$\mathbf{x} \to \mathbf{x} + \mathbf{\Delta x} \; t \to t + \Delta t \; h \to h + \Delta h \; \mathbf{x} \to -\mathbf{x} \; h \to -h, \tag{265}$$

we are led to the Edwards-Wilkinson equation (Edwards and Wilkinson, 1982),

$$\partial_t h(\mathbf{x},t) = v + \sigma \nabla^2 h(\mathbf{x},t) + \zeta(\mathbf{x},t), \tag{266}$$

which is the simplest stochastic differential equation that describes a surface growth with these symmetries. Here v denotes the mean growth velocity, σ the surface tension, and ζ the zero-average Gaussian noise field with variance

$$\langle \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t')\rangle = 2D\,\delta^{d-1}(\mathbf{x}=\mathbf{x}')(t-t'). \tag{267}$$

This equation is linear and exactly solvable. The critical exponents of Edwards-Wilkinson classes are

$$\tilde{\beta} = \left(\frac{1}{2} - \frac{d}{4}\right), \quad Z=2. \tag{268}$$

C. Quenched Edwards-Wilkinson classes

In random media, linear interface growth is described by the so-called quenched Edwards-Wilkinson equation,

$$\partial_t h(\mathbf{x},t) = \sigma \nabla^2 h(\mathbf{x},t) + F + \eta(\mathbf{x},h(\mathbf{x},t)), \tag{269}$$

where F is a constant, external driving term and $\eta(\mathbf{x},h(\mathbf{x},t))$ is the quenched noise. The corresponding linear interface models exhibit a depinning transition at F_c. The universal behavior of these models was investigated by Nattermann *et al.* (1992); Narayan and Fisher (1993); Kim *et al.* (2001), and it was shown to be equivalent to the nondiffusive conserved-field (NDCF) classes (Sec. V.I). Analytical studies (Narayan and Fisher, 1993) predict $\tilde{\alpha}=(4-d)/3$ and $Z=2-(2/9)(4-d)$.

D. Kardar-Parisi-Zhang classes

If we drop the $h \to -h$ symmetry from Eq. (265) we can add a term to Eq. (266) that is the most relevant term in the renormalization sense, breaking the up-down symmetry:

$$\partial_t h(\mathbf{x},t) = v + \sigma \nabla^2 h(\mathbf{x},t) + \lambda [\nabla h(\mathbf{x},t)]^2 + \zeta(\mathbf{x},t). \tag{270}$$

This is the Kardar-Parisi-Zhang equation (Kardar *et al.*, 1986). Here again v denotes the mean growth velocity, σ the surface tension, and ζ the zero-average Gaussian noise field with variance

$$\langle \zeta(\mathbf{x},t)\zeta(\mathbf{x}',t')\rangle = 2D\,\delta^{d-1}(\mathbf{x}-\mathbf{x}')(t-t'). \tag{271}$$

This is nonlinear, but exhibits a tilting symmetry as a result of the Galilean invariance of Eq. (270):

$$h \to h' + \epsilon \mathbf{x}, \quad \mathbf{x} \to \mathbf{x}' - \lambda \epsilon t, \quad t \to t', \tag{272}$$

where ϵ is an infinitesimal angle. As a consequence the scaling relation

$$\tilde{\alpha} + Z = 2 \tag{273}$$

holds in any dimensions. In one dimension the critical exponents are known exactly, whereas for $d>1$ dimensions numerical estimates exist (Barabási and Stanley, 1995; see Table XXVII). The upper critical dimension of this model is debated. Mode-coupling theories and various phenomenological field-theoretical schemes (Halpin-Healy, 1990; Lässig, 1995; Lässig and Kinzelbach, 1997) settle to $d_c=4$. In contrast to analytical approaches, numerical solutions of the Kardar-Parisi-Zhang equation (Wolf and Kertész, 1987), simulations (Kim and Kosterlitz, 1989; Moser *et al.*, 1991; Tang *et al.*, 1992; Ala-Nissila *et al.*, 1993; Marinari *et al.*, 2000), and the results of real-space renormalization-group calcula-

TABLE XXVII. Scaling exponents of the Kardar-Parisi-Zhang model from Barabási and Stanley (1995).

d	$\tilde{\alpha}$	$\tilde{\beta}$	Z
1	1/2	1/3	3/2
2	0.38	0.24	1.58
3	0.30	0.18	1.66

tions (Castellano, Gabrielli, *et al.*, 1998, 1999; Castellano, Marsili, and Pietronero, 1998) provide no evidence for a finite d_c. Furthermore, the only numerical study (Tu, 1994) of the mode-coupling equations gives no indication of the existence of a finite d_c either. Recently simulations of restricted solid-on-solid growth models were used to build the width distributions of two- to five-dimensional Kardar-Parisi-Zhang interfaces. The universal scaling function associated with the steady-state width distribution was found to change smoothly as d was increased, thus strongly suggesting that $d=4$ is not an upper critical dimension for the Kardar-Parisi-Zhang equation. The dimensional trends observed in the scaling functions indicate that the upper critical dimension is at infinity (Marinari *et al.*, 2002).

In the hope of classifying nonequilibrium phase-transition classes according to their noise terms, Grinstein, Munoz, and Tu (1996) and Tu *et al.* (1997) introduced and studied systems via the Langevin equation

$$\partial_t n(x,t) = D\nabla^2 n(x,t) - rn(x,t) - un(x,t)^2 + n(x,t)\eta(x,t), \tag{274}$$

exhibiting real multiplicative noise:

$$\langle \eta(x,t)\rangle = 0, \quad \langle \eta(x,t)\eta(x',t')\rangle = 2\nu\delta^d(x-x')\delta(t-t'), \tag{275}$$

which is proportional to the field (n). The form of the Langevin noise term can be deduced from a field-theoretic action for the system derived from a microscopic master equation (Cardy, 1997). For the DP class the noise is real, while for ARW and parity-conserving systems the noise is imaginary. Howard and Täuber (1997) investigated two of the simplest reaction-diffusion systems in which both real and imaginary noise are present and compete: (a) $2A\to\emptyset$, $2A\to 2B$, $2B\to 2A$, and $2B\to\emptyset$; (b) $2A\to\emptyset$ and $2A\to(n+2)A$ (see Sec. V.F). In neither case did they recover the multiplicative noise critical behavior reported by Grinstein, Munoz, and Tu (1996) or Tu *et al.* (1997). Therefore they suspected that there might not be a real reaction-diffusion system possessing the multiplicative noise behavior.

On the other hand, Grinstein, Munoz, and Tu (1996) and Tu *et al.* (1997) have established a connection between multiplicative noise systems and the Kardar-Parisi-Zhang theory via the Cole-Hopf transformation: $n(x,t)=e^{h(x,t)}$. They have shown in one dimension that the phase diagram and the critical exponents Z, $\nu_\perp$, and β of the two systems agree within numerical accuracy. They found diverging susceptibility (with continuously changing exponents as a function of r) for the entire range of r.

E. Quenched Kardar-Parisi-Zhang classes

In random media nonlinear interface growth is described by the so-called *quenched Kardar-Parisi-Zhang equation* (Barabási and Stanley, 1995),

$$\partial_t h(\mathbf{x},t) = \sigma\nabla^2 h(\mathbf{x},t) + \lambda[\nabla h(\mathbf{x},t)]^2 + F + \eta(\mathbf{x},h(\mathbf{x},t)), \tag{276}$$

where F is a constant, external driving term and $\eta(\mathbf{x},h(\mathbf{x},t))$ is the quenched noise (which does not fluctuate in time). The universal behavior of this equation was investigated by Buldyrev *et al.* (1993) and Leschorn (1996), who predicted $\tilde{\alpha}\simeq 0.63$ in one dimension, $\tilde{\alpha}\simeq 0.48$ in two dimensions, and $\tilde{\alpha}\simeq 0.38$ in three dimensions. It was shown numerically that in one dimension this class is described by (1+1)-dimensional directed-percolation depinning (Tang and Leschhorn, 1992). In higher dimensions, however, it is related to percolating directed surfaces (Barabási *et al.*, 1996).

F. Other continuum growth classes

For continuum growth models exhibiting the symmetries

$$\mathbf{x}\to\mathbf{x}+\Delta\mathbf{x}, \quad t\to t+\Delta t, \quad h\to h+\Delta h, \quad \mathbf{x}\to -\mathbf{x}, \tag{277}$$

there are several possible general Langevin equations, as follows (Barabási and Stanley, 1995):

- The deterministic part of the equation may describe a conservative or nonconservative process (i.e., the integral over the entire system may be zero or not). The conservative terms are $\nabla^2 h$, $\nabla^4 h$, and $\nabla^2(\nabla h)^2$. The only relevant nonconservative term is the $(\nabla h)^2$.
- The system may be linear or not.
- The noise term may be conservative (i.e., the result of some surface diffusion) with correlator

$$\langle \zeta_d(\mathbf{x},t)\zeta_d(\mathbf{x}',t')\rangle = (-2D_d\nabla^2 + D_d'\nabla^4)\delta(\mathbf{x}-\mathbf{x}')\delta(t-t'), \tag{278}$$

or it may be nonconservative as in Eq. (267), as the result of adsorption-desorption mechanisms.

When the surface growth properties of systems other than the Edwards-Wilkinson and Kardar-Parisi-Zhang classes were analyzed, five other universality classes were identified (see Table XXVIII).

G. Models of mass adsorption-desorption, aggregation, and chipping

Based on the interest in self-organized critical systems in which different physical quantities exhibit power-law distributions in the steady state over a wide region of the

TABLE XXVIII. Summary of continuum growth classes discussed in this section, following Barabási and Stanley (1995).

Langevin equation	$\tilde{\alpha}$	$\tilde{\beta}$	Z
$\partial_t h = -K\nabla^4 h + \zeta$	$(4-d)/2$	$(4-d)/8$	4
$\partial_t h = \nu\nabla^2 h + \zeta_d$	$-d/2$	$-d/4$	2
$\partial_t h = -K\nabla^4 h + \zeta_d$	$(2-d)/2$	$(2-d)/8$	4
$\partial_t h = -K\nabla^4 h + \lambda_1\nabla^2(\nabla h)^2 + \zeta$	$(4-d)/3$	$(4-d)/8+d$	$(8+d)/3$
		$d\leqslant 1$	
$\partial_t h = -K\nabla^4 h + \lambda_1\nabla^2(\nabla h)^2 + \zeta_d$	$(2-d)/3$	$(2-d)/10+d$	$(10+d)/3$
		$d>1$	
	$(2-d)/2$	$(2-d)/8$	4

parameter space (Bak *et al.*, 1987), a family of lattice models in which masses diffuse, aggregate on contact, or chip off a single unit mass was introduced by Majumdar *et al.* (1998, 2000a). Self-organized criticality has been studied in a variety of model systems ranging from sandpiles to earthquakes. A particularly simple lattice model due to Takayasu, with mass diffusion, aggregation upon contact, and adsorption of unit masses from outside at a constant rate, exhibits self-organized criticality (Takayasu *et al.*, 1988; Takayasu and Takayasu, 1997): the steady-state mass distribution has a nontrivial power-law decay for large masses in all dimensions (Takayasu *et al.*, 1988). These mass adsorption-desorption models in one dimension are defined as follows. A site i is chosen randomly and then one of the following events can occur:

(1) Adsorption: With rate q, a single particle is adsorbed at site i; thus $m_i \to m_i+1$.

(2) Desorption: With rate p, a single particle is desorbed from site i; thus $m_i \to m_i - 1$ provided $m_i \geqslant 1$.

(3) Diffusion and aggregation: With rate 1, the mass m_i at site i moves to a nearest-neighbor site [either $(i-1)$ or $(i+1)$] chosen at random. If it moves to a site that already has some particles, then the total mass just adds up; thus $m_i \to 0$ and $m_{i\pm1} \to m_{i\pm1} + m_i$.

(4) Chipping: With rate w a bit of mass at the site "chips" off, e.g., provided $m_i \geqslant 1$, a single particle leaves site i.

While the Takayasu model $(p=0, w=0)$ does not have a phase transition in the steady state, by introducing a nonzero desorption rate p one can induce a critical line $p_c(q)$ in the p-q plane. For fixed q, if one increases p from 0, one finds that for all $p < p_c(q)$, the steady-state mass distribution has the same large-m behavior as in the Takayasu case, i.e.,

$$P(m) \propto m^{-\tau}, \tag{279}$$

where the exponent τ is the Takayasu exponent and is independent of q. For $p = p_c(q)$, we find that the steady-state mass distribution still decays algebraically for large m, but with a new critical exponent τ_c which is bigger than the Takayasu exponent τ_t. For $p > p_c(q)$, we find that

$$P(m) \sim \exp(-m/m^*) \tag{280}$$

for large m, where m^* is a characteristic mass that diverges if one approaches $p_c(q)$ from the $p > p_c(q)$ side. The critical exponent τ_c is the same everywhere on the critical line $p_c(q)$. This phase transition occurs in all spatial dimensions, including $d=1$. The τ exponent was determined for the mean-field and one-dimensional cases (Takayasu *et al.*, 1988; Majumdar *et al.*, 2000b),

$$\tau_t^{MF} = 3/2, \quad \tau_c^{MF} = 5/2, \quad \tau_t^{1d} = 4/3, \quad \tau_c^{1d} = 1.833, \tag{281}$$

although the location of d_c is not known.

This model can also be mapped onto interface dynamics, if we interpret the configuration of masses as an interface profile, regarding m_i as a local height variable. The phase transition of the model can be qualitatively interpreted as a nonequilibrium wetting transition of the interface. The corresponding surface growth exponents are (Majumdar *et al.*, 2000b)

$$\tilde{\beta}^{MF} = 1/6, \quad Z=2, \quad \tilde{\beta}^{1d} = 0.358, \quad Z=2. \tag{282}$$

In the $p=q=0$ conserved model, a nonequilibrium phase transition occurs when the chipping rate or the average mass per site ρ is varied. There is a critical line $\rho_c(w)$ in the ρ-w plane that separates two types of asymptotic behaviors of $P(m)$. For fixed w, as ρ is varied across the critical value $\rho_c(w)$, the large m behavior of $P(m)$ was found to be

$$P(m) \sim \begin{cases} e^{-m/m^*}, & \rho < \rho_c(w), \\ m^{-\tau}, & \rho = \rho_c(w), \\ m^{-\tau} + \text{infinite aggregate}, & \rho > \rho_c(w). \end{cases} \tag{283}$$

As one increases ρ beyond ρ_c, this asymptotic algebraic part of the critical distribution remains unchanged but in addition an infinite aggregate forms. This means that all the additional mass $(\rho-\rho_c)V$ (where V is the volume of the system) condenses onto a single site and does not disturb the background critical distribution. This is analogous, in spirit, to the condensation of a macroscopic number of bosons onto the single $k=0$ mode in an ideal Bose gas as the temperature goes below a certain critical value. Rajesh and Majumdar (2001) proved analytically that the mean-field phase boundary, $\rho_c(w) = \sqrt{w+1} - 1$, is exact and independent of the spatial dimension d. They also provided unambiguous numerical evidence that the exponent $\tau = 5/2$ is also independent of d. The corresponding growth exponents are $Z=2$ and $\tilde{\alpha} = 2/3$ (Majumdar *et al.*, 2000a). Even though the single-site distribution $P(m)$ may be given exactly by the mean-field solution, that does not prove that mean-field theory or product measure is the exact stationary state in all dimensions.

The left-right asymmetric version of the conserved model was also studied by Majumdar *et al.* (2000a). This has a qualitatively similar phase transition in the steady state to that of the conserved model but exhibits a different class phase transition owing to the mass current in this system. Simulations in one dimension predict $Z=1.67$, $\tilde{\alpha}=0.67$.

H. Unidirectionally coupled DP classes

As was mentioned in Sec. V.J in the case of coupled, multispecies directed-percolation processes, field-theoretical RG analysis (Janssen, 1997b) predicts DP criticality with an unstable, symmetrical fixed point, such that subsystems with unidirectionally coupled DP behavior emerge. This was shown to be a valid prediction for linearly coupled, N-component, contact processes too. Unidirectionally coupled DP systems of the form

$$A \leftrightarrow 2A, \quad A \to A + B,$$

$$B \leftrightarrow 2B, \quad B \to B + C,$$

$$C \leftrightarrow 2C, \quad C \to C + D \tag{284}$$

were investigated by Täuber, Howard, and Hinrichsen (1998) and Goldschmidt *et al.* (1999) with the motivation that such models can describe interface growth models, where adsorption-desorption are allowed at terraces and edges (see Sec. VI.H). The simplest set of Langevin equations for such systems was set up by Alon *et al.* (1998):

$$\partial_t \phi_k(\mathbf{x},t) = \sigma \phi_k(\mathbf{x},t) - \lambda \phi_k^2(\mathbf{x},t) + D\nabla^2 \phi_k(\mathbf{x},t) + \mu \phi_{k-1}(\mathbf{x},t) + \zeta_k(\mathbf{x},t), \tag{285}$$

where ζ_k are independent multiplicative noise fields for level k with the correlations

$$\langle \zeta_k(\mathbf{x},t) \rangle = 0, \tag{286}$$

$$\langle \zeta_k(\mathbf{x},t) \zeta_l(\mathbf{x}',t') \rangle = 2\Gamma \phi_k(\mathbf{x},t) \delta_{k,l} \delta^d(\mathbf{x}-\mathbf{x}') \delta(t-t'),$$

for $k>0$, while for the lowest level ($k=0$) $\phi_{-1}\equiv 0$ is fixed. The parameter σ controls the offspring production, μ is the coupling, and λ is the coagulation rate. As one can see, the $k=0$ equation is just the Langevin equation for directed percolation (84). The mean-field solution of these equations that is valid above $d_c=4$ results in critical exponents for the level k:

$$\beta_{MF}^{(k)} = 2^{-k} \tag{287}$$

and $\nu_{\perp}^{MF}=1/2$ and $\nu_{\parallel}^{MF}=1$ independently of k. For $d<d_c$ field-theoretical RG analysis of the action for $k<K$ levels

TABLE XXIX. Critical exponents of unidirectionally coupled directed percolation. From Goldschmidt *et al.*, 1994.

	$d=1$	$d=2$	$d=3$	$d=4-\epsilon$
β_1	0.280(5)	0.57(2)	0.80(4)	$1-\epsilon/6+O(\epsilon^2)$
β_2	0.132(15)	0.32(3)	0.40(3)	$1/2-\epsilon/8+O(\epsilon^2)$
β_3	0.045(10)	0.15(3)	0.17(2)	$1/4-O(\epsilon)$
δ_1	0.157(4)	0.46(2)	0.73(5)	$1-\epsilon/4+O(\epsilon^2)$
δ_2	0.075(10)	0.26(3)	0.35(5)	$1/2-\epsilon/6+O(\epsilon^2)$
δ_3	0.03(1)	0.13(3)	0.15(3)	$1/4-O(\epsilon)$
η_1	0.312(6)	0.20(2)	0.10(3)	$\epsilon/12+O(\epsilon^2)$
η_2	0.39(2)	0.39(3)	0.43(5)	$1/2+O(\epsilon^2)$
η_3	0.47(2)	0.56(4)	0.75(10)	$3/4-O(\epsilon)$
$2/Z_1$	1.26(1)	1.10(2)	1.03(2)	
$2/Z_2$	1.25(3)	1.12(3)	1.04(2)	$1+\epsilon/24+O(\epsilon^2)$
$2/Z_3$	1.23(3)	1.10(3)	1.03(2)	
$\nu_{\perp,1}$	1.12(4)	0.70(4)	0.57(4)	
$\nu_{\perp,2}$	1.11(15)	0.69(15)	0.59(8)	$1/2+\epsilon/16+O(\epsilon^2)$
$\nu_{\perp,3}$	0.95(25)	0.65(15)	0.62(9)	
$\nu_{\parallel,1}$	1.78(6)	1.24(6)	1.10(8)	
$\nu_{\parallel,2}$	1.76(25)	1.23(17)	1.14(15)	$1+\epsilon/12+O(\epsilon^2)$
$\nu_{\parallel,3}$	1.50(40)	1.15(30)	1.21(15)	

$$S = \sum_{k=0}^{K-1} \int d^d x dt \Big\{ \psi_k(\tau\partial_t - D\nabla^2 - \sigma)\phi_k - \mu \psi_k \phi_{k-1} + \frac{\Gamma}{2}\psi_k(\phi_k - \psi_k)\phi_k \Big\} \tag{288}$$

was performed by Täuber, Howard, and Hinrichsen (1998) and Goldschmidt *et al.* (1999). The RG treatment runs into several difficulties. Infrared-divergent diagrams were encountered (Goldschmidt, 1998) and the coupling constant μ was shown to be a relevant quantity (which means it diverges under RG transformations). Goldschmidt *et al.* (1999) argued that this is the reason why scaling seems to break down in simulations for large times (in lattice realization μ is limited). The exponents of the one-loop calculations for the first few levels, corresponding to the interactive fixed line, as well as results of lattice simulations, are shown in Table XXIX. These scaling exponents can be observed for intermediate times, but it is not clear whether in an asymptotically long time they drift to the decoupled values or not.

The main representatives of these classes are certain monomer adsorption-desorption models (Sec. VI.H) and polynuclear growth models with depinning transitions (Kertész and Wolf, 1989; Lehner *et al.*, 1990; Toom, 1994a, 1994b). The latter types of systems are defined by parallel update dynamic rules, and coupled DP processes emerge in a co-moving frame.

A coupled particle system can be related to an interface growth model. Alon *et al.* (1996, 1998) defined solid-on-solid (SOS) and restricted solid-on-solid (RSOS) models that can be mapped onto unidirectionally coupled directed percolation (Sec. VI.H). In these models adsorption or desorption processes may take place at terraces and edges. For each update a site i is chosen at random and an atom is adsorbed

$$h_i \to h_i + 1 \text{ with probability } q \tag{289}$$

or desorbed at the edge of a plateau

$$h_i \to \min(h_i,h_{i+1}) \text{ with probability } (1-q)/2,$$

$$h_i \to \min(h_i,h_{i-1}) \text{ with probability } (1-q)/2. \tag{290}$$

Identifying empty sites at a given layer as A particles, the adsorption process can be interpreted as the decay of A particles ($A \to \emptyset$), while the desorption process corresponds to A-particle production ($A \to 2A$). These processes generate reactions on subsequent layers, hence they are coupled. The simulations in one dimension have shown that this coupling is relevant in the upward direction only, hence the model is equivalent to unidirectionally coupled directed percolation. Defining the order parameters on the kth layer as

$$n_k = \frac{1}{N}\sum_i \sum_{j=0}^{k} \delta_{h_i,j}, \tag{291}$$

where h_i is the height at site i, they are expected to scale as

$$n_k \sim (q_c - q)^{\beta^{(k)}}, \quad k = 1,2,3,\ldots. \tag{292}$$

When the growth rate (q) is varied, these models exhibit a roughening transition at q_c=0.189 (for RSOS) and at q_c=0.233(1) (for SOS) from a nonmoving, smooth phase to a moving, rough phase in one spatial dimension. The β^k (and other exponents) take those of the one-dimensional unidirectionally coupled DP class (see Table XXIX).

The scaling behavior of the interface width is characterized by many different length scales. At criticality it increases as

$$W^2(t) \propto \tau \ln t \tag{293}$$

with $\tau \simeq 0.102(3)$ for RSOS (Hinrichsen and Mukamel, 2003).

I. Unidirectionally coupled parity-conserving classes

Surface growth processes of dimers stimulated the introduction of unidirectionally coupled BARW2 (Sec. IV.D.1) models (Hinrichsen and Ódor, 1999a, 1999b):

$$A \to 3A, \quad B \to 3B, \quad C \to 3C,$$

$$2A \to \emptyset, \quad 2B \to \emptyset, \quad 2C \to \emptyset, \tag{294}$$

$$A \to A+B, \quad B \to B+C, \quad C \to C+D, \quad \ldots$$

generalizing the concept of unidirectionally coupled directed percolation (see Sec. VI.H). The mean-field approximation of the reaction scheme (294) looks like

$$\partial_t n_A = \sigma n_A - \lambda n_A^2,$$

$$\partial_t n_B = \sigma n_B - \lambda n_B^2 + \mu n_A, \tag{295}$$

$$\partial_t n_C = \sigma n_C - \lambda n_C^2 + \mu n_B, \quad \ldots$$

where n_A, n_B, n_C correspond to the densities n_0, n_1, n_2 in the growth models. Here σ and λ are the rates for offspring production and pair annihilation, respectively. The coefficient μ is an effective coupling constant between different particle species. Since these equations are coupled in only one direction, they can be solved by iteration. Obviously, the mean-field critical point is $\sigma_c = 0$. For small values of σ the stationary particle densities in the active state are given by

$$n_A = \frac{\sigma}{\lambda}, \quad n_B \simeq \frac{\mu}{\lambda}\left(\frac{\sigma}{\mu}\right)^{1/2}, \quad n_C \simeq \frac{\mu}{\lambda}\left(\frac{\sigma}{\mu}\right)^{1/4}, \tag{296}$$

corresponding to the mean-field critical exponents

$$\beta_A^{MF} = 1, \quad \beta_B^{MF} = 1/2, \quad \beta_C^{MF} = 1/4, \quad \ldots. \tag{297}$$

These exponents should be valid for $d > d_c = 2$. Solving for asymptotic temporal behavior one finds $\nu_\| = 1$, implying that $\delta_k^{MF} = 2^{-k}$.

The effective action of unidirectionally coupled BARW2's should be given by

$$S[\psi_0,\psi_1,\psi_2,\ldots,\bar\psi_0,\bar\psi_1,\bar\psi_2,\ldots]$$
$$= \int d^dx dt \sum_{k=0}^{\infty} \{\bar\psi_k(\partial_t - D\nabla^2)\psi_k - \lambda(1-\bar\psi_k^2)\psi_k^2$$
$$+ \sigma(1-\bar\psi_k^2)\bar\psi_k\psi_k + \mu(1-\bar\psi_k)\bar\psi_{k-1}\psi_{k-1}\}, \tag{298}$$

where $\psi_{-1} = \bar\psi_{-1} \equiv 0$. Here the fields ψ_k and $\bar\psi_k$ represent the configurations of the system at level k. Since even the RG analysis of the one-component BARW2 model suffered serious problems (Cardy and Täuber, 1996), the solution of the theory of Eq. (298) seems to be hopeless. Furthermore one expects IR diagram problems and diverging coupling strengths, as in the case of unidirectional coupling. These might be responsible for violations of scaling in the long-time limit.

Simulations of a three-component model in one dimension, coupled by instantaneous particle production of the form (294), resulted in decay exponents for the order parameter defined as (291):

$$\delta_A = 0.280(5), \quad \delta_B = 0.190(7), \quad \delta_C = 0.120(10). \tag{299}$$

For further critical exponents see Sec. VI.I.

It would be interesting to investigate parity-conserving growth processes in higher dimensions. Since the upper critical dimension d_c' is less than 2, one expects the roughening transition—if one still exists—to be described by mean-field exponents. In higher dimensions, n-mers might appear in different shapes and orientations.

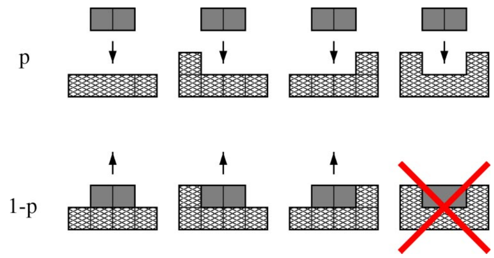

FIG. 27. (Color in online edition) Absorption of dimers with probability p and desorbtion at the edges of terraces with probability $1-p$. Evaporation from the middle of plateaus is not allowed. From Hinrichsen and Ódor, 1999a.

The main representatives of these classes are certain dimer adsorption-desorption models (Sec.VI.H) and polynuclear growth models with depinning transitions (Hinrichsen and Ódor, 1999a). The latter types of systems are defined by parallel update dynamic rules, and coupled DP processes emerge in a comoving frame.

Similarly to the monomer case (Sec. VI.H), dimer adsorption-desorption models were defined (Hinrichsen and Ódor, 1999a, 1999b; Noh *et al.*, 2000). With the restriction that desorption may only take place at the edges of a plateau, the models can be mapped onto the unidirectionally coupled BARW2 class (Sec. VI.I). The dynamical rules in $d=1$ are defined in Fig. 27. The mapping onto unidirectionally coupled BARW2 can be seen in Fig. 28.

Such dimer models can be defined in arbitrary spatial dimensions. Hinrichsen and Ódor (1999a) investigated four one-dimensional variants:

(1) Variant A is a restricted solid-on-solid model evolving by random sequential updates.

(2) Variant B is a solid-on-solid model evolving by random sequential updates.

(3) Variant C is a restricted solid-on-solid model evolving by parallel updates.

(4) Variant D is a solid-on-solid model evolving by parallel updates.

Variants A and B exhibit transitions in contrast with polynuclear growth models, in which only parallel update rules permit roughening transitions. When the adsorption rate p is varied, the phase diagram shown in Fig. 29 emerges for RSOS and SOS cases. If p is very small, only a few dimers are adsorbed at the surface, staying there for a short time before they evaporate back into the gas phase. Thus the interface is anchored to the actual bottom layer and does not propagate. In this smooth phase the interface width grows logarithmically until it saturates at a finite value (even for $L\to\infty$).

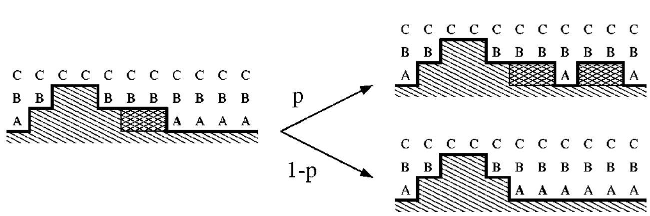

FIG. 28. Extended particle interpretation. Dimers are adsorbed ($2A\to\emptyset$) and desorbed ($A\to 3A$) at the bottom layer. Similar processes take place at higher levels. From Hinrichsen and Ódor, 1999a.

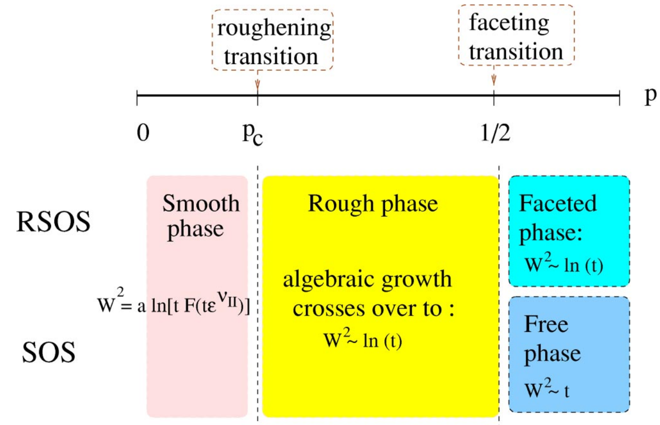

FIG. 29. (Color in online edition) Phase diagram of one-dimensional dimer models.

As p increases, a growing number of dimers covers the surface, and large islands of several layers stacked on top of each other are formed. Approaching a certain critical threshold p_c the mean size of the islands diverges and the interface evolves into a rough state with the finite-size scaling form

$$W^2(L,t) \simeq a \ln[tG(t/L^Z)]. \tag{300}$$

The order parameter defined on the kth layer as Eq. (294) exhibits unidirectionally coupled BARWe critical behavior. The transition rates and exponents are summarized in Table XXX. Above p_c one may expect the interface to detach from the bottom layer in the same way as the interface of monomer models starts to propagate in the supercritical phase. However, since dimers are adsorbed at neighboring lattice sites, solitary unoc-

TABLE XXX. Numerical estimates for the four variants of the dimer model at the roughening transition $p=p_c$ (upper part) and at the transition $p=0.5$ (lower part).

Variant	A	B	C	D
Restriction updates	yes random	no random	yes parallel	no parallel
p_c	0.3167(2)	0.292(1)	0.3407(1)	0.302(1)
a	0.172(5)	0.23(1)	0.162(4)	0.19(1)
Z	1.75(5)	1.75(5)	1.74(3)	1.77(5)
δ_0	0.28(2)	0.29(2)	0.275(10)	0.29(2)
δ_1	0.22(2)	0.21(2)	0.205(15)	0.21(2)
δ_2	0.14(2)	0.14(3)	0.13(2)	0.14(2)
$\tilde{\alpha}$	1.2(1)	undefined	1.25(5)	undefined
$\tilde{\beta}$	0.34(1)	0.50(1)	0.330(5)	0.49(1)

TABLE XXXI. Summary of known absorbing-state universality classes in homogeneous isotropic systems: DP, directed percolation; DyP, dynamical percolation; VM, voter model; PCP, pair-contact process; NDLF, nondiffusive conserved field; PC, parity-conserving; BP, binary production; DCF, diffusive conserved field; N-BARW2, N-component branching and annihilating random walk with two offspring; N-BARW2s, symmetric N-BARW2; N-BARW2a, asymmetric N-BARW2.

CLASS	Features	Section
DP	time-reversal symmetry	IV.A
DyP	long memory	IV.B
VM	Z_2 symmetry	IV.C
PCP	coupled frozen field	V.E
NDCF	global conservation	V.I
PC	Z_2 symmetry, BARW2 conservation	IV.D.1
BP	DP coupled to ARW	V.F
DCF	coupled diffusive conserved field	V.H
N-BARW2	N-component BARW2 conservation	V.K
N-BARW2s	symmetric NBARW2+exclusion	V.L
N-BARW2a	asymmetric NBARW2+exclusion	V.L

cupied sites may emerge. These pinning centers prevent the interface from moving and lead to the formation of droplets. Due to interface fluctuations, the pinning centers can slowly diffuse to the left and to the right. When two of them meet at the same place, they annihilate, and a larger droplet is formed. Thus, although the interface remains pinned, its roughness increases continuously. The width initially increases algebraically until it slowly crosses over to a logarithmic increase $W(t)\sim\sqrt{a\ln t}$.

The restricted as well as the unrestricted variants undergo a second phase transition at $p=0.5$ (Noh *et al.*, 2000), where the width increases algebraically with time as $W\sim t^{\tilde{\beta}}$. In the RSOS case ordinary Family-Vicsek scaling [Eq. (254)] occurs, with the exponents given in Table XXX. The dynamic exponent is $Z=\tilde{\alpha}/\tilde{\beta}\simeq 3$. This value stays the same if one allows dimer digging at the faceting transition, but other surface exponents $\tilde{\alpha}\simeq 0.29(4)$ and $\tilde{\beta}\simeq 0.111(2)$ will be different (Noh *et al.*, 2000). An explanation of the latter exponents is given by Noh *et al.* (2001) based on mapping to globally constrained random walks. In the SOS cases (variants B,D), large spikes are formed, and the surface roughens much faster with a growth exponent of $\tilde{\beta}\simeq 0.5$. The interface evolves into configurations with large columns of dimers separated by pinning centers. These spikes can grow or shrink almost independently. As the columns are spatially decoupled, the width does not saturate in finite systems, i.e., the dynamic exponents $\tilde{\alpha}$ and Z have no physical meaning.

For $p>0.5$ the restricted models A and C evolve into faceted configurations. The width first increases algebraically until the pinning centers become relevant and the system crosses over to a logarithmic increase in the width. Therefore the faceted phase may be considered as a rough phase. The unrestricted models B and D, however, evolve into spiky interface configurations. The spikes are separated and grow independently by deposition of dimers. Therefore W^2 increases linearly with time, defining the free phase of the unrestricted models.

In the simulations mentioned up until now the interface was grown from flat initial conditions. It turns out that when one starts with random initial conditions $h_i=0,1$ the densities n_k turn out to decay much more slowly. For restricted variants an algebraic decay of n_0 with an exponent

$$\delta_0\simeq 0.13 \tag{301}$$

was observed. Similarly, the critical properties of the faceting transition at $p=0.5$ are affected by random initial conditions. The nonuniversal behavior for random initial conditions is related to an additional parity conservation law. The dynamic rules not only conserve parity of the particle number but also conserve the parity of droplet sizes. Starting with a flat interface, the lateral

TABLE XXXII. Mean-field classes of known, homogeneous absorbing-state transitions: DP, directed percolation; DyP, dynamic percolation; VM, voter model; PC, parity conserving; BKARW, branching and k-particle annihilating random walk; PARWs, symmetric production and m-particle annihilating random walk; PARWa, asymmetric production and m-particle annihilating random walk; NDCF, nondiffusive conserved field; NBARW2, N-component branching and annihilating random walk with even offspring.

CLASS	β	β'	Z	$\nu_\parallel$	α	δ	η	d_c
DP	1	1	2	1	1	1	0	4
DyP	1	1	2	1	1	1	0	6
VM	0	1	2	1	0	1	0	2
PC	1	0	2	1	1	0	$-1/2$	2
BkARW	$1/(k-1)$	0	2	1	$1/(k-1)$	0	0	$2/(k-1)$
PARWs	1	0	2	n	$1/n$	0	0	
PARWa	$1/(m-n)$		2	$(m-1)/(m-n)$	$1/(m-1)$			
NDCF	1	1	2	1	1	1	0	4
NBARW2	1	0	2	1	1	0	0	2

size of droplets is always even, allowing them to evaporate entirely. However, for a random initial configuration, droplets of odd size may be formed which have to recombine in pairs before they can evaporate, slowing down the dynamics of the system. In the language of BARW2 processes, the additional parity conservation is due to the absence of nearest-neighbor diffusion. Particles can only move by a combination of offspring production and annihilation, i.e., by steps of *two* lattice sites. Therefore particles at even and odd lattice sites have to be distinguished. Only particles of different parity can annihilate. Starting with a fully occupied lattice all particles have alternating parity throughout the whole temporal evolution, leading to the usual critical behavior at the parity-conserving transition. For random initial conditions, however, particles of equal parity cannot annihilate, slowing down the decay of the particle density. Similar sector decomposition has been observed in diffusion of k-mer models (Barma *et al.*, 1993; Barma and Dhar, 1994).

VII. SUMMARY

In summary, dynamical extensions of classical equilibrium classes were introduced in the first part of this review. New exponents, concepts, subclasses, mixing dynamics, and some unresolved problems were discussed. The common behavior of these models was the strongly fluctuating ordered state. In the second part of the review genuine nonequilibrium dynamical classes of reaction-diffusion systems and interface growth models were discussed. These were related to phase transitions to absorbing states of weakly fluctuating ordered states. The behavior of these classes is usually determined by the spatial dimensions, symmetries, boundary conditions, and inhomogeneities, as in the case of the equilibrium models, but in low dimensions hard-core exclusion was found to be a relevant factor too, making the criticality in fermionic and bosonic models different. The symmetries are not so evident as in case of equilibrium models. They are most precisely expressed in terms of the relations of fields and response fields. Furthermore, in recently discovered coupled systems with binary, triplet, or quadruplet particle production no special symmetry seems to be responsible for a novel type of critical behavior. Perhaps a proper field-theoretical analysis of the coupled DP+ARW system could shed some light on this mystery. Parity conservation in hard-core and in binary spreading models seems to be irrelevant. Table XXXI summarizes the best-known families of absorbing phase transition classes of homogeneous, spatially isotropic systems. Those which are in the lower part exhibit *fluctuating absorbing states*. The necessary and sufficient conditions for these classes are usually unknown. The mean-field classes can also give a guide to distinguishing classes below d_c. Table XXXII collects the mean-field exponents and upper critical dimensions of the known absorbing-state model classes. Note that in the general $nA\to(n+k)A$, $mA\to(m-l)A$ type of RD systems the values of n and m determine the critical behavior.

In $d>1$ dimension the mapping of spin-systems onto RD systems of particles is not straightforward; instead one should also take into account the theory of branching interfaces (Cardy, 2000). Preliminary simulations using generalized Potts models have found interesting critical phenomena exhibiting absorbing states (Lipowski and Droz, 2002a).

Further research is needed to explore the universality classes of nonequilibrium phase transitions that are induced by external current and of other models exhibiting fluctuating ordered states (Evans *et al.*, 1998; Evans, 2000). Nonequilibrium phase transitions in quantum systems (Rácz, 2002) or in irregular graph or network-based systems are also a current interest of research. Finally, having settled the problems raised by fundamental nonequilibrium models, one should turn towards the study of more applied systems.

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LIST OF ABBREVIATIONS

ABC	active boundary condition	IV.A.4
ARW	annihilating random walk ($AA\to\emptyset$)	IV.C.1
BARW	branching and annihilation random walk	IV.A
BARWe	even-offspring branching and annihilation random walk	IV.D.1
BARWo	odd-offspring branching and annihilation random walk	IV.A.3
BARW2	two-offspring branching and annihilation random walk	IV.D.1
BkARW	branching and k particle annihilating random walk	IV.E
BP	binary production	V.F
CAM	coherent anomaly method	IV.A
CDP	compact directed percolation	IV.A.2
DCF	diffusive conserved field	V.H
DK	Domany-Kinzel (cellular automaton)	IV.A.2
DP	directed percolation	IV.A
DS	damage spreading	I.D
DyP	dynamical percolation	IV.B
EW	Edwards-Wilkinson	VI.B
DMRG	density matrix renormalization group	IV
GEP	generalized epidemic process	IV.B
GDK	generalized Domany-Kinzel (cellular automaton)	IV.D.3

GMF	generalized mean-field approximation	II
IBC	inactive boundary condition	IV.D.4
IMD	interacting monomer-dimer	IV.A.4
KPZ	Kardar-Parisi-Zhang	VI.D
LIM	linear interface model	VI.C
MF	mean-field approximation	I.E
MM	monomer-monomer	IV.D.4
N-BARW2	even-offspring, *N*-component branching and annihilation random walk	V.K
NDCF	nondiffusive conserved field	V.I
NEKIM	nonequilibrium Ising model	IV.D.2
PC	parity conserving	IV.D
PCP	pair contact process	V.E
PCPD	pair contact process with particle diffusion	V.F.1
PARWs	symmetric production and *m*-particle annihilating random walk	IV.F
PARWa	asymmetric production and *m*-particle annihilating random walk	IV.F
PNG	polinuclear growth models	VI.H
RBC	reflecting boundary condition	IV.A.4
RD	reaction-diffusion	I.E
RG	renormalization group	I.E
RSOS	restricted solid on solid model	IV
SCA	stochastic cellular automaton	IV.A.2
SOS	solid on solid model	VI
TTP	threshold transfer process	V.E
UCDP	unidirectionally coupled directed percolation	VI.H
VM	voter model	IV.C

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