Slow dynamics on quenched complex networks

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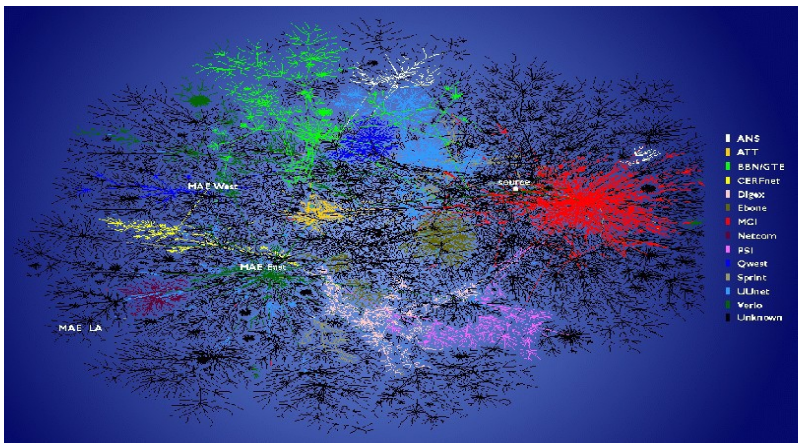
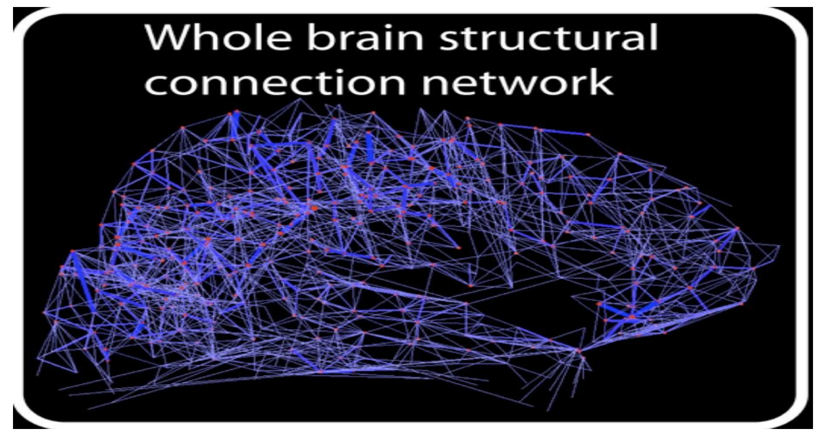
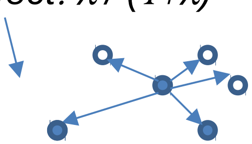
Statphys research $\rightarrow$ dynamical processes defined on complex networks
Expectation: small world topology $\rightarrow$ mean-field behavior $\rightarrow$ fast dynamics



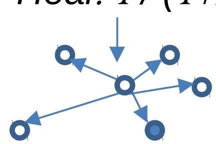


*Prototype: **C**ontact **P**rocess (**CP**) or **S**usceptible-**I**nfected-**S**usceptible (**SIS**) two-state models:*

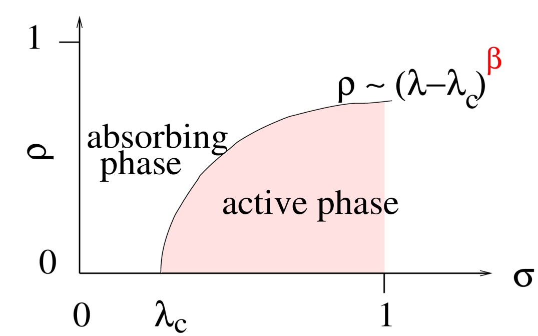
Infect: $\lambda / (1+\lambda)$



Heal: $1 / (1+\lambda)$



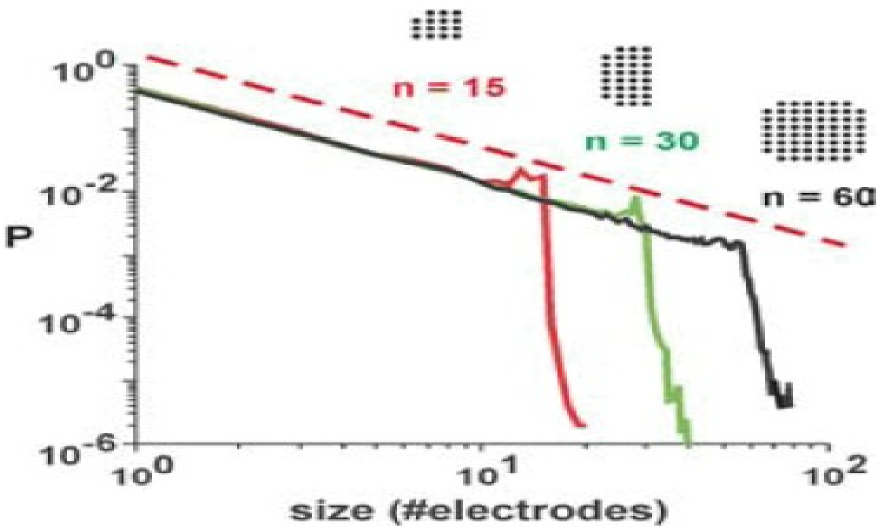
Order parameter : density of active (●) sites
*Regular, Euclidean lattice: **DP** critical point : $\lambda_c > 0$ between inactive and active phases*



Observed power-law (slow) dynamics in networks

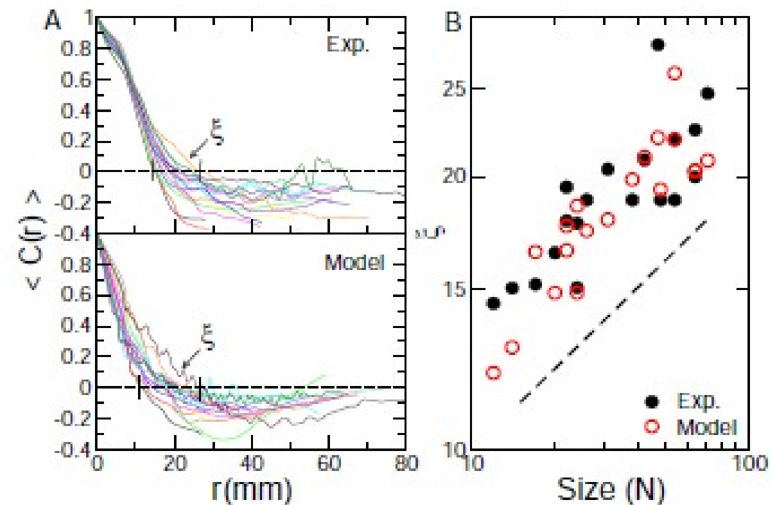
- Brain : Size distribution of neural avalanches

G. Werner : Biosystems, 90 (2007) 496,

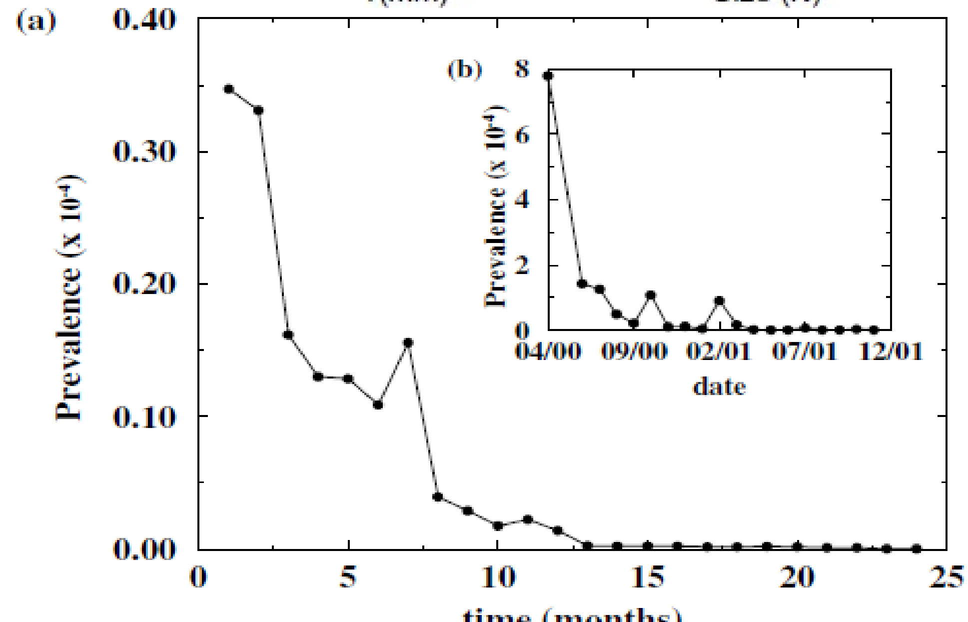


- *Correlation length (ξ) diverges*

Tagliazucchi & Chialvo (2012) :
Brain complexity born out of criticality.



Internet: worm recovery time is slow:



Can we expect slow dynamics in small-world networks ?

Scaling in nonequilibrium system

Scaling and universality classes appear in complex system due to : $\xi \to \infty$
i.e: near critical points, due to currents ...

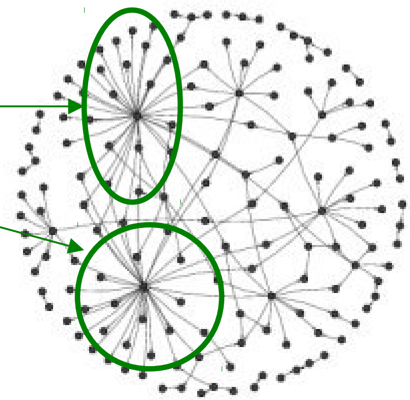
Basic models are classified by universal scaling behavior in Euclidean, regular system

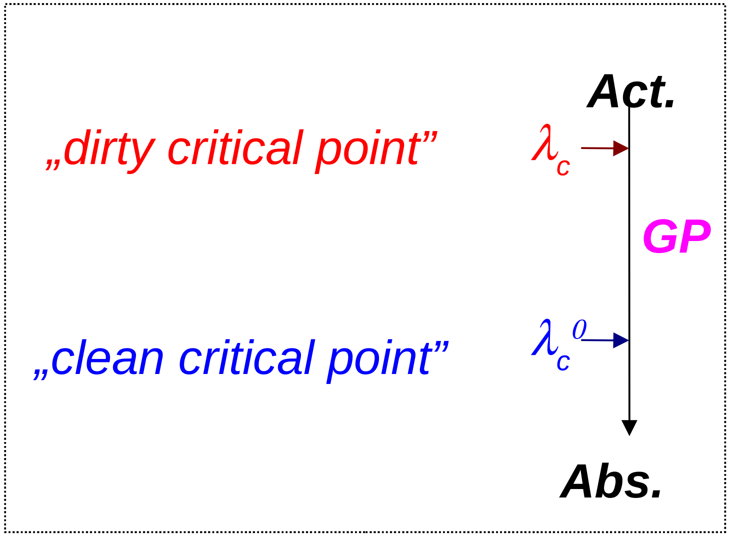
- **Why don't we see universality classes in models defined on networks ?**
- **Power laws are frequent in nature $\leftrightarrow$ Tuning to critical point ?**

I'll show a possible way to understand these

R**are **R**egion theory for quench disordered **CP

- *Fixed (quenched) disorder/impurity*
  changes the local birth rate $\Rightarrow \lambda_c > \lambda_c^0$
- ***Locally active**, but arbitrarily large*
  Rare Regions
  in the inactive phase
  due to the inhomogeneities





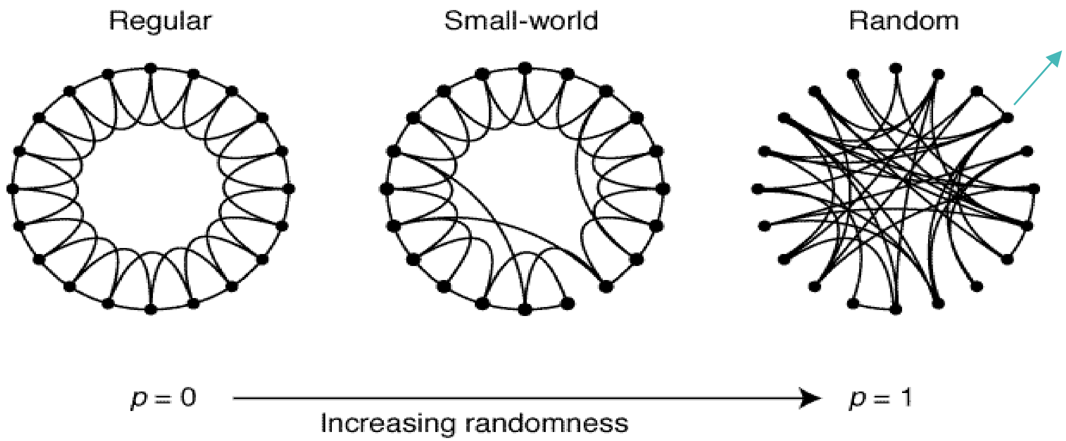
- *Probability of RR of size L_R:*
  $w(L_R) \sim \exp(-c\, L_R)$

contribute to the density: $\rho(t) \sim \int dL_R\; L_R\; w(L_R)\; \exp[-t/\tau(L_R)]$

- ***For*** $\lambda < \lambda_c^0$ *: conventional (exponentially fast) decay*
- ***At*** λ_c^0 *the characteristic time scales as:* $\tau(L_R) \sim L_R^{\,Z} \Rightarrow$ *saddle point analysis:*
  $\ln \rho(t) \sim t^{d/(d+Z)}$ *stretched exponential*
- ***For*** $\lambda_c^0 < \lambda < \lambda_c$ *:* $\tau(L_R) \sim \exp(b\, L_R)$*:* ***Griffiths Phase***
  $\rho(t) \sim t^{-c/b}$ *continuously changing exponents*
- ***At*** λ_c $\rho(t) \sim \ln(t)^{-\alpha}$ *Infinite randomness fixed point scaling*
- ***In case of correlated RR-s with dimension*** $> d^-$ ***: smeared transition***

Basic network models

From regular to random networks:

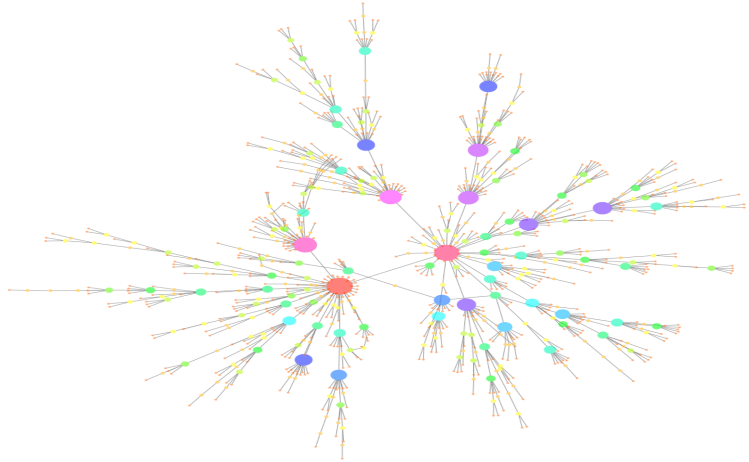


Erdős-Rényi $(p = 1)$

Degree (k) distribution in $N \rightarrow \infty$ node limit:
$P(k) = e^{-<k>} <k>^k / k!$

Topological dimension: N(r) ~ r ^{d}
Above perc. thresh.: $d = \infty$
Below percolation $d = 0$

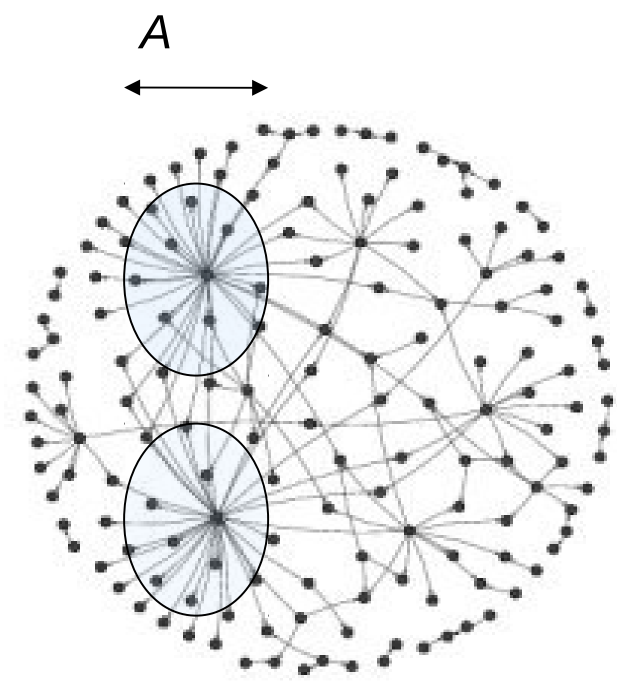
Scale free networks:



Degree distribution:
$P(k) = k^{-\gamma} \; (2 < \gamma < 3)$

Topological dimension: $d = \infty$

Example: **Barabási-Albert lin. prefetential attachment**



Rare active regions in the absorbing phase: $\tau(A) \sim e^{A}$
→ slow dynamics (Griffiths Phase) ?

M. A. Munoz, R. Juhász, C. Castellano and G. Ódor, PRL 105, 128701 (2010)

1. Inherent disorder in couplings
2. Disorder induced by topology

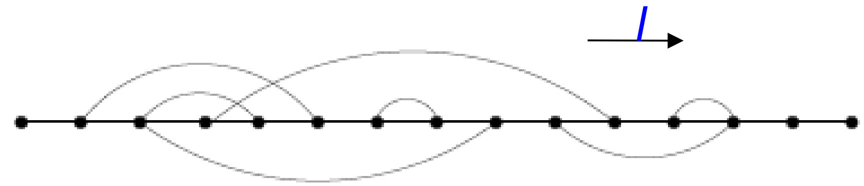
Optimal fluctuation theory + simulations: YES

- In Erdős-Rényi networks below the percolation threshold
- In generalized small-world networks for finite topological dimension

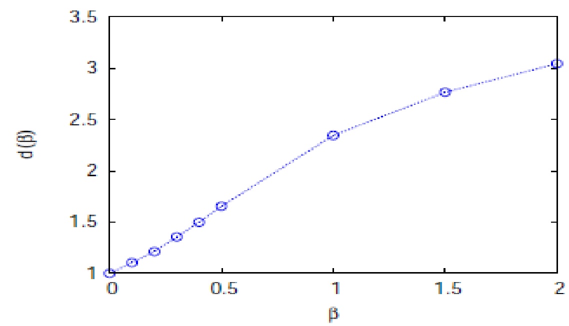
CP + Topological disorder results

Generalized Small World networks: $P(l) \sim \beta\, l^{-2}$

(link length probability)



Top. dim: $N(r) \sim r^{d}$ $d(\beta)$ **finite:**

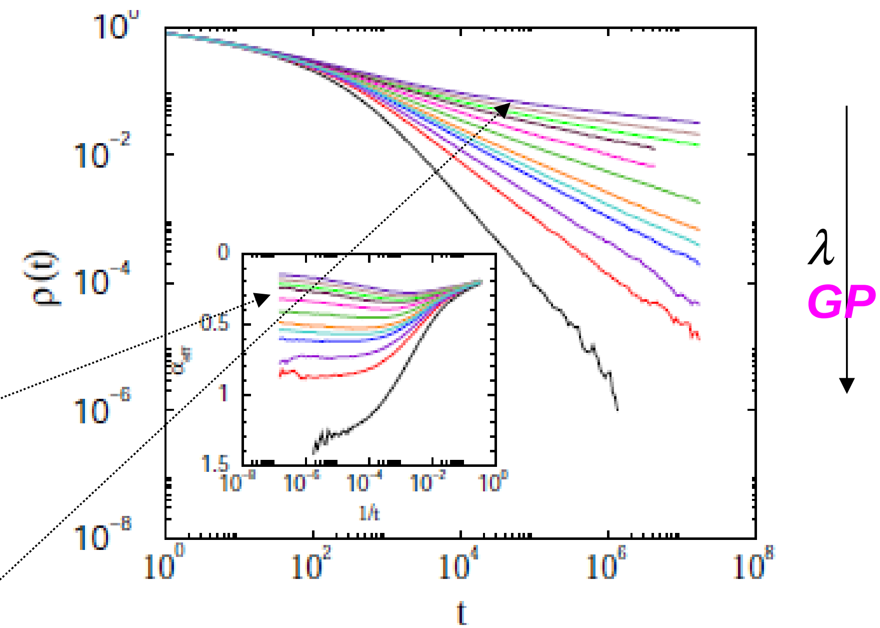


$\lambda_c(\beta)$ *decreases* monotonically from
$\lambda_c(0) = 3.29785$ *(1d* **CP***) to:*
$\lim_{\beta \to \infty} \lambda_c(\beta) = 1$ towards mean-field **CP** value

$\lambda < \lambda_c(\beta)$ *:* inactive, there can be
locally ordered, **rare regions** due to more than average, active, incoming links

Griffiths phase: *λ – dep. continuously changing dynamical power laws:*
for example : $\rho(t) \sim t^{-\alpha(\lambda)}$

Logarithmic corrections !



Ultra-slow (“activated”) scaling: $\rho \propto \ln(t)^{-\alpha}$ at λ_c

As $\beta \to 1$ Griffiths phase shrinks/disappears

Same results for: cubic, regular random nets
higher dimensions ?

FIG. 3: Density decay in Benjamini-Berger networks with $s = 2$ and $\beta = 0.2$ for different values of λ (from top to bottom: $2.81, 2.795, 2.782, 2.77, 2.75, 2.73, 2.71, 2.70, 2.69, 2.67, 2.65, 2.6$). Straight lines lie in the Griffiths phase. Inset: Corresponding effective exponents, illustrating the presence of corrections to scaling.

Contact Process on Barabási-Albert (BA) network

- Heterogeneous mean-field theory: conventional critical point, with linear density decay:

$$\rho(t) \sim [t \ln(t)]^{-1},$$

with logarithmic correction

- Extensive simulations confirm this:
- No Griffiths phase observed
- Steady state density vanishes at $\lambda_c \approx 1$ linearly, HMF: $\beta = 1$

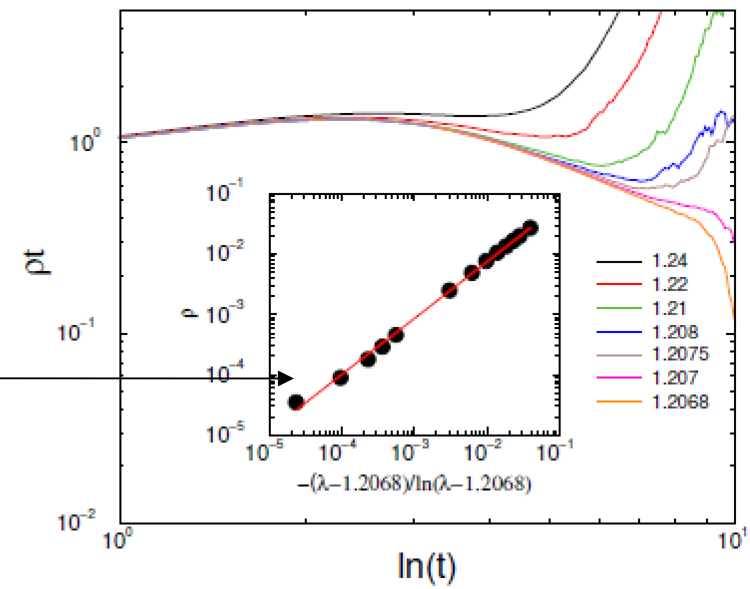


FIG. 1. Density decay ($t\rho(t)$) as a function of $\ln(t)$ for the CP on unweighted looped BA networks with $m = 3$ of size $N = 8 \times 10^7$. The different curves correspond to $\lambda = 1.2068, ..., 1.24$ (bottom to top). Inset: Steady state density, showing agreement with HMF theory scaling. The full line shows a power-law fitting to the data points in the form $-0.36(5)x^{0.98(2)}$.

CP results on Barabási-Albert graphs

- *Excluding loops slows down the spreading* + *Weights:*

  G. Ódor, R. Pastor-Storras PRE 86 (2012) 026117

$$\omega_{ij} = \frac{|i-j|^x}{N} \qquad k_i \propto (N/i)^{1/2}$$

- *WBAT-I:* $\omega_{ij} = \omega_0 (k_i k_j)^{-\nu}$ *hubs are supressed*

WBAT-II: *disassortative*

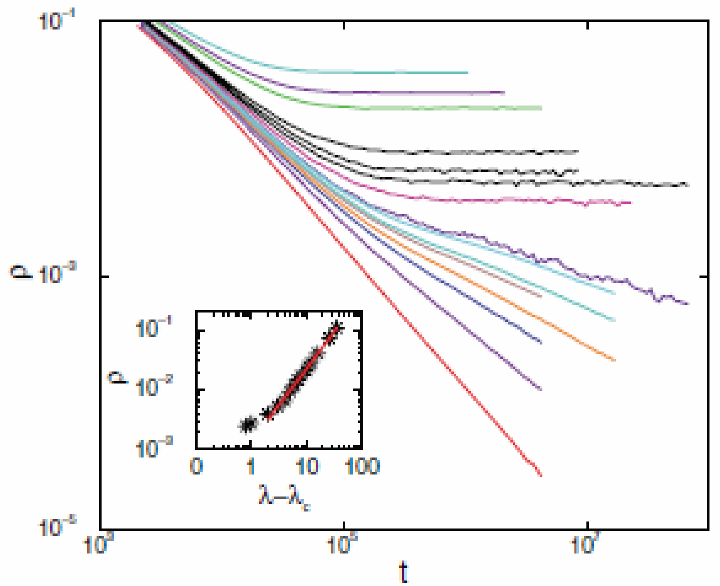


FIG. 3. (Color online) Density decay as a function of time for the CP on weighted BA trees generated with the WBAT-I model with exponent $\nu = 1.5$. Network size $N = 10^5$. Different curves correspond to $\lambda = 160, 156, 154, 149, 148, 147, 146, 145, 144.7, 144.2, 144, 143.5, 143, 142, 140$ (from top to bottom). Inset: Steady-state density.

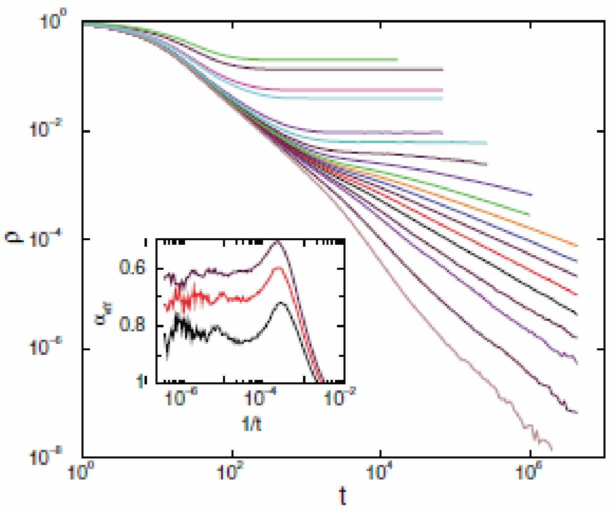


FIG. 7. (Color online) Density decay as a function of time $\rho(t)$ for the CP on weighted BA trees with a age-dependent weighting scheme (WBAT-II) with exponent $x = 2$. Network size $N = 10^5$. Different curves correspond to $\lambda = 6.75, 6.8, 6.85, 6.87, 6.9, 6.92, 6.94, 6.96, 6.98, 7, 7.04, 7.1, 7.2, 7.4, 8.5, 9, 12, 15$ (from top to bottom). Inset: Corresponding local slopes for $\lambda = 6.9, 6.92, 6.94$ (from bottom to top).

λ dependent density decay exponents: Griffiths Phases or Smeared phase transition ?

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Do power-laws survive the thermodynamic limit ?

- Finite size analysis shows the disappearance of a power-law scaling:

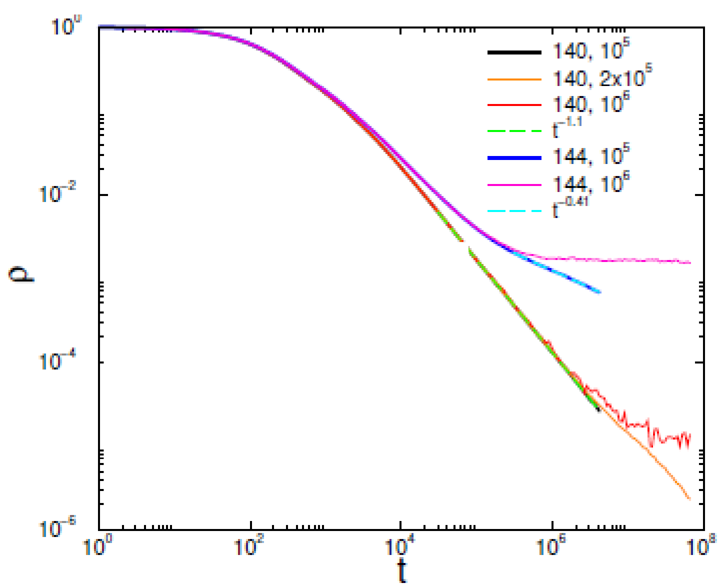


FIG. 5. Density decay as a function of time $\rho(t)$ for the CP on weighted BA trees with a multiplicative weighting scheme (WBAT-I) with exponent $\nu = 1.5$. Plots correspond to two sets of λ (upper branch: $\lambda = 144$, lower branch $\lambda = 140$) at different network sizes N. Dashed lines represent PL fittings. Inset: Initial time region of the same data, showing an stretched exponential behavior.

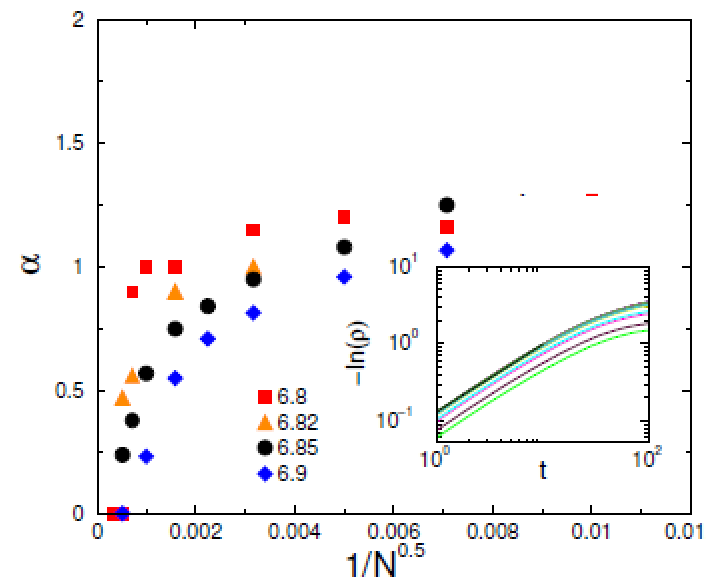


FIG. 8. Finite-size scaling analysis of the density decay exponent for $\lambda = 6.75$ (triangles), $\lambda = 6.8$ (boxes), $\lambda = 6.82$ (triangles), $\lambda = 6.85$ (bullets), $\lambda = 6,9$ (rhombes) in the CP on weighted BA trees with a age-dependent weighting scheme (WBAT-II) with exponent $x = 2$. Top inset: $\rho(t)$ for $\lambda = 6.82$ ($N = 10^6$, $N = 4x10^5$, $N = 10^5$ top to bottom). Bottom inset: Initial time density.

- Power-law → saturation explained by smeared phase transition:
  High dimensional rare sub-spaces

Percolation analysis of the weighted BA tree

We consider a network of a given size N, and delete all the edges with a weight smaller than a threshold ω_{th}.

For small values of ω_{th}, many edges remain in the system, and they form a connected network with a single cluster encompassing almost all the vertices in the network.

When increasing the value of ω_{th}, the network breaks down into smaller subnetworks of connected edges, joined by weights larger than ω_{th}.

The size of the largest ones ($\boldsymbol{S_i}$) grows linearly with the network size N

↔ standard percolation transition.

These clusters, which can become arbitrarily large in the thermodynamic limit, play the role of correlated **RR**s, sustaining independently activity and smearing down the phase transition.

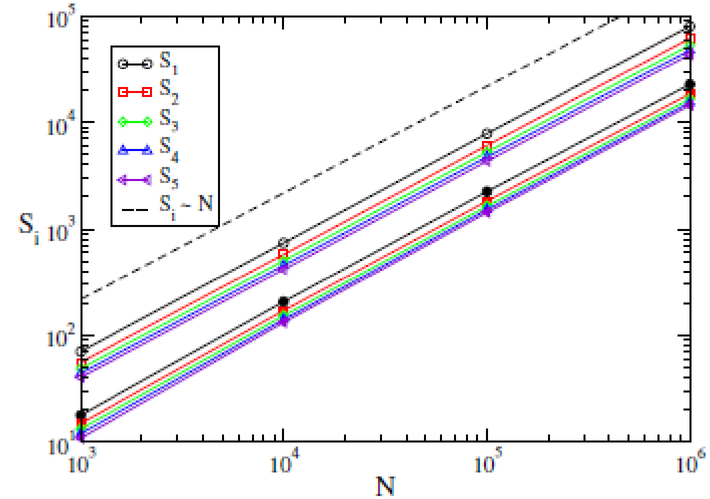


FIG. 6. Size S_i of the 5 largest clusters in a percolation analysis of the WBAT-I model with $\nu = 1.5$ for $\omega_{th} = 100\omega_{min}$ (hollow symbols) and $\omega_{th} = 1000\omega_{min}$ (full symbols), where ω_{min} is the minimum weight in the network. The size of all components grows linearly with network size N, and is therefore infinite in the thermodynamic limit.

Spectral Analysis of networks with quenched (disordered) topology

Master (rate) equation of SIS for occupancy prob. at site i:

$$\frac{d\rho_i(t)}{dt} = -\rho_i(t) + (1-\rho_i(t))\sum_j A_{ij}\lambda\rho_j(t)\ \omega_{ij} \quad (4)$$

For $t \to \infty$ the system evolves into a steady state, with the probabilities expressed as

$$\rho_i = \frac{\lambda \sum_j B_{ij}\rho_j}{1+\lambda\sum_j B_{ij}\rho_j}\,. \quad (5)$$

Weighted (real symmetric) Adjacency matrix: $B_{ij} = A_{ij}\omega_{ij},$

Express ρ_i *on orthonormal eigenvector (* $\mathbf{f_i(\Lambda)}$ *) basis:*

$$\rho_i = \sum_\Lambda c(\Lambda) f_i(\Lambda). \quad (6)$$

$$c(\Lambda) = \lambda \sum_{\Lambda'} \Lambda' c(\Lambda') \sum_{i=1}^{N} \frac{f_i(\Lambda) f_i(\Lambda')}{1+\lambda\sum_{\tilde{\Lambda}} \tilde{\Lambda} c(\tilde{\Lambda}) f_i(\tilde{\Lambda})}. \quad (7)$$

$\lambda_c = 1/\Lambda_1$ $\quad\quad$ $\Lambda_1(N) \propto N^{1/4}$

Total infection density vanishes near λ_c *as :*

$$\rho(\lambda) \approx \alpha_1\tau + \alpha_2\tau^2 + \dots\,, \quad (8)$$

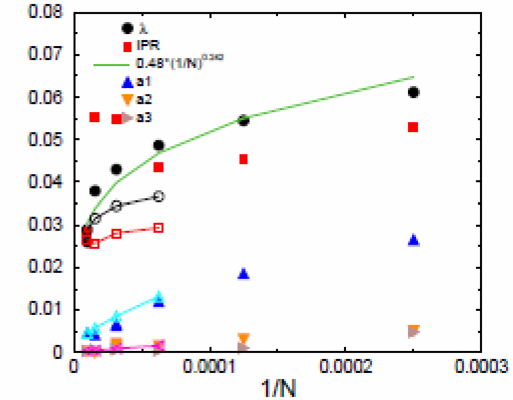
where $\tau = \lambda\Lambda_1 - 1 \ll 1$ with the coefficients

$$\alpha_j = \sum_{i=1}^{N} f_i(\Lambda_j)/[N\sum_{i=1}^{N} f_i^3(\Lambda_j)]. \quad (9)$$

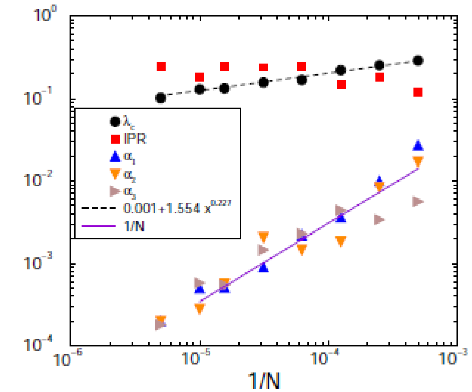
Suggested by Goltsev, Dorogovtschev & Mendes 2012

To describe the localization of the components of $\boldsymbol{f}(\Lambda_1)$ [19] used the inverse participation ratio

$$IPR(\Lambda) \equiv \sum_{i=1}^{N} f_i^4(\Lambda), \quad (10)$$



Unweighted BAT



WBAT-II

Network	$1/\Lambda_1$	c	IPR	α_1	α_2	α_3
BA	0.017	0.32	0.014	4×10^{-4}	10^{-7}	10^{-8}
BAT	~ 0	0.24	0.055	10^{-4}	10^{-7}	10^{-9}
WBAT-I	~ 0	0.9(1)	~ 0	0.35(1)	5×10^{-8}	10^{-6}
WBAT-II	0.001	0.227	0.25	4×10^{-3}	3×10^{-5}	10^{-3}

Localization, strong rare-region effects in case of WBAT-II networks !

Summary

- *Quenched disorder in complex networks can cause slow dynamics :*
  Rare-regions → (Griffiths) phasess → no tuning or self-organization needed !

- *In **finite dim**. (for CP) GP can occur due to topological disorder*

- *In **infinite dim**, scale-free, BA network mean-field transition of CP with logarithmic corrections (HMF+simulations)*

- *In BA **trees** non mean-field transition observed*

- *In **weighted** BA **trees** non-universal, slow, power-law dynamics can occur for finite N, but in the N $\to\infty$ limit saturation is observed*

- *Smeared transition can describe this,*
  percolation analysis confirms the existence of arbitrarily large dimensional sub-spaces with (correlated) large weights

- *Acknowledgements to : HPC-Europa2, OTKA, FuturICT.hu*

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