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Orthogonal elementary interactions for bimatrix games

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ABSTRACT

In symmetric matrix games, the interaction is defined through a single payoff matrix that can be decomposed into elementary interactions of four types representing games with self- and cross-dependent payoffs, coordination-type interactions, and rock–paper–scissors-like cyclic dominance. Here, this analysis is extended to a similar anatomy of bimatrix games given by two matrices. The respective self- and cross-dependent components describe extended versions of donation games expanding the range of social dilemmas. The most attractive classification utilizes the fact that games can be separated into the sum of a fraternal and a zero-sum part whose payoff matrices can then be further divided into symmetric and antisymmetric terms. This approach revealed two types of interaction not present in symmetric games: directed anticoordination components share some features with both partnership games and rock–paper–scissors-like cyclic dominance; the combinations of matching pennies components prevent the existence of a potential, which precludes detailed balance with the Boltzmann distribution under Glauber-type dynamics. In another departure from symmetric games, bimatrix games may admit a non-Hermitian potential matrix, which could possibly give rise to thermodynamic behaviors not found in classical spin models. Some curiosities of the directed anticoordination interaction are illustrated by simulations when the players are located on a square lattice.

1. Introduction

Evolutionary game theory has already become a general mathematical framework for the study of a wide range of phenomena in social, biological, information, and other non-equilibrium systems [1–13]. In a family of these mathematical models, players are distributed on the sites of a lattice or a network and they can each choose one of n strategies in games played with their neighbors. For each player x, the accumulated benefit $u_x(s_x, s_{-x})$ depends on their own choice, s_x, and also on their coplayers' strategies, collectively denoted here in shorthand notation as s_{-x}. For most evolutionary games, pair interactions are defined by a uniform, symmetric, two-player n-strategy game and the evolution of the strategy distribution s (using the notations introduced above, for any focal player x, $s = \{s_x, s_{-x}\}$) is controlled by a dynamical rule that may involve personal features, preferences, external effects, etc. After a transient period, these systems evolve into a final, stationary state that is determined by the specific pair interaction(s), connectivity structure, system size, dynamical rule(s), and initial state of the population and can be characterized via the averages of the strategy frequencies. A quantity that often plays a key role in determining the properties of this final state is called potential [14–16]: A game admits the potential v if for all x players and all $\{s'_x, s_{-x}\}$ and $\{s_x, s_{-x}\}$ strategy profile pairs the condition

$$v(\{s'_x, s_{-x}\}) - v(\{s_x, s_{-x}\}) = u_x(s'_x, s_{-x}) - u_x(s_x, s_{-x}) \quad (1)$$

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is satisfied, that is, if a single v function exists that contains the payoff difference incentives of every possible unilateral strategy change of all players as differences between the function's values at the corresponding strategy profiles. This game potential defined on the abstract space of strategy profiles (when equipped with a topology derived from unilateral strategy changes) is analogous to the potential of a conservative force field in real space.

In this paper, our efforts are focused on the classification of pair interactions [17–22] that fundamentally affect the appearance of thermodynamical behavior, social dilemmas, and the formation of self-organizing patterns in games in which two types of players (female or male, young or old, small or large, etc.) can be distinguished. In the next section, we briefly survey the concept of matrix decomposition for symmetric two-player n-strategy games for reference. This approach exploits the fact that a single matrix can be considered as a linear combination of orthogonal basis matrices representing four types of elementary interactions [23,24]. Selecting these orthogonal elementary interactions is consistent with the isomorphism [25,26] identifying games (interactions or matrices) that can be transformed into each other by a permutation of the strategy labels. More precisely, the four types of elementary interactions exhibit proper features and symmetries that are preserved when the strategy labels are exchanged.

For non-symmetric games, the number of payoff parameters (the dimension of the parameter space) is doubled, which gives rise to new types of interactions. For example, symmetric two-player, two-strategy games ($n = 2$) always admit a potential because the low degree of freedom prohibits the appearance of a cyclic component, which precludes the existence of an otherwise always symmetric potential matrix in symmetric games with more strategies. In contrast, the existence of a potential can be prevented by the presence of a matching pennies component even in $n = 2$ bimatrix games, and asymmetric (non-Hermitian) potential matrices can be derived. We have to emphasize that potential games [14,15] form a subset of interactions whose multiplayer evolutionary games become equivalent to many-particle systems – i.e., Ising- or Potts-type models, described well by the concepts and tools of statistical physics [27–30] – when the players are located at the sites of a lattice and follow a suitable dynamical rule. Very recently the investigation of many-particle models has been extended to have non-reciprocal pair interactions occurring in living systems [31] and evolutionary games [16].

In the next section, we discuss the concept and results of matrix decomposition for symmetric games, using a notation that clearly visualizes the differences between the four types of orthogonal elementary interactions. In Section 3 this formalism is extended to the investigation of bimatrix games that reveals new types of interaction not present for symmetric games. The consequences of the most interesting one are illustrated by considering a simple spatial evolutionary game having a non-Hermitian potential in Section 4. Finally, the novelties and perspectives are summarized.

2. A brief survey of the decomposition of symmetric games

For a symmetric two-player n-strategy game, the pair interaction is defined by an $n \times n$ payoff matrix $\mathbf{A}$ whose elements determine the players' payoffs for all possible strategy pairs $[i,j]$ ($i,j = 1,2,\ldots,n$). In this notation A_{ij} is a real number that quantifies the benefit of the first player if she chooses the ith strategy while her coplayer selects the jth strategy that provides a payoff A_{ji} for the second player [32–35]. In these games, the players are equivalent in that they receive the same payoff (A_{ii}) if they both choose the ith strategy and their payoffs are exchanged ($A_{ij} \leftrightarrow A_{ji}$) if they exchange their strategies ($[i,j] \leftrightarrow [j,i]$).

The payoff matrix can be considered as a linear combination of Cartesian-type orthogonal basis matrices containing a single 1 while their other entries are 0. In this approach, the expansion coefficients of these basis matrices are given by the A_{ij} payoffs. Two matrices (e.g., $\mathbf{A}$ and $\tilde{\mathbf{A}}$) are orthogonal if their scalar product is zero, that is, if $\mathbf{A} \cdot \tilde{\mathbf{A}} = \sum_{i,j} A_{ij} \tilde{A}_{ij} = 0$. Instead of the Cartesian-type basis matrices one can choose other sets of basis matrices reflecting some universal features. Now let us briefly survey a matrix decomposition that illustrates the existence of four fundamentally different types of interactions [23,24]. In this framework, the payoff matrix is built up from four plus one types of orthogonal components. Namely,

$$\mathbf{A} = a\mathbf{U} + \mathbf{A}^{(\mathrm{se})} + \mathbf{A}^{(\mathrm{cr})} + \mathbf{A}^{(\mathrm{co})} + \mathbf{A}^{(\mathrm{cyc})}\,, \tag{2}$$

where the first term,

$$a\mathbf{U} = a\begin{pmatrix} 1 & 1 & \ldots & 1 \\ 1 & 1 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & 1 \end{pmatrix}, \text{ with } a = \frac{1}{n^2}\sum_{j,k} A_{jk}\,, \tag{3}$$

measures the average value of the payoffs, which typically represents an irrelevant constant contribution with respect to incentives to unilateral strategy changes. The self-dependent payoff part of a game is defined by a matrix with identical elements in its rows as

$$\mathbf{A}^{(\mathrm{se})} = \begin{pmatrix} \varepsilon_1 & \varepsilon_1 & \ldots & \varepsilon_1 \\ \varepsilon_2 & \varepsilon_2 & \ldots & \varepsilon_2 \\ \vdots & \vdots & \ddots & \vdots \\ \varepsilon_n & \varepsilon_n & \ldots & \varepsilon_n \end{pmatrix}, \text{ where } \varepsilon_i = \frac{1}{n}\sum_j A_{ij} - a\,. \tag{4}$$

Similarly, the cross-dependent payoff part is given by $\mathbf{A}^{(\mathrm{cr})}$ with equivalent entries in its columns as

$$\mathbf{A}^{(\mathrm{cr})} = \begin{pmatrix} \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \\ \vdots & \vdots & \ddots & \vdots \\ \gamma_1 & \gamma_2 & \cdots & \gamma_n \end{pmatrix}, \text{ where } \gamma_j = \frac{1}{n}\sum_i A_{ij} - a\,. \tag{5}$$

These components account for average payoff values when playing or playing against a fixed strategy, respectively, and their contributions to the payoffs are determined by the strategy choice of only one of the players. The above definitions of $\mathbf{A}^{(\mathrm{se})}$ and $\mathbf{A}^{(\mathrm{cr})}$ take into consideration the identical entries in the rows and columns of $a\mathbf{U}$. The orthogonality conditions $\mathbf{U}\cdot\mathbf{A}^{(\mathrm{se})} = \mathbf{U}\cdot\mathbf{A}^{(\mathrm{cr})} = \mathbf{A}^{(\mathrm{se})}\cdot\mathbf{A}^{(\mathrm{cr})} = 0$ are satisfied because $\sum_i \epsilon_i = \sum_j \gamma_j = 0$. These conditions reduce the number of independent parameters (dimensions) defining the sum of the average, self-, and cross-dependent components mentioned above to 1, $(n-1)$, and $(n-1)$, respectively. The permutation of strategy labeling leaves the main characteristics of these components unchanged.

In the matrix $\mathbf{R} = \mathbf{A} - a\mathbf{U} - \mathbf{A}^{(\mathrm{se})} - \mathbf{A}^{(\mathrm{cr})}$, the sum of payoffs is zero in every row and column, in agreement with the orthogonality conditions. This matrix can be decomposed into the sum of its symmetric $[\mathbf{A}^{(\mathrm{co})} = (\mathbf{R}+\mathbf{R}^{+})/2]$ and antisymmetric $[\mathbf{A}^{(\mathrm{cyc})} = (\mathbf{R}-\mathbf{R}^{+})/2]$ parts, which satisfy the orthogonality condition $\mathbf{A}^{(\mathrm{co})}\cdot\mathbf{A}^{(\mathrm{cyc})} = 0$.

The matrix $\mathbf{A}^{(\mathrm{co})}$ summarizes the contributions of elementary coordination components $\mathbf{Q}^{(\mathrm{k,l})}$ for all possible strategy pairs $[k,l]$ $(k<l)$ as

$$\mathbf{A}^{(\mathrm{co})} = \sum_{\substack{k,l\\k<l}} \nu_{kl}\mathbf{Q}^{(\mathrm{k,l})}, \tag{6}$$

where $\nu_{kl} = -A^{(\mathrm{co})}_{kl}$. The $\mathbf{Q}^{(\mathrm{k,l})}$ matrices are sparse, containing only two diagonal entries with value $+1$ and two opposite non-diagonal -1 entries. For example,

$$\mathbf{Q}^{(1,2)} = \begin{pmatrix} +1 & -1 & 0 & \dots & 0 \\ -1 & +1 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \tag{7}$$

and the other $\mathbf{Q}^{(\mathrm{k,l})}$ basis matrices can be obtained from $\mathbf{Q}^{(1,2)}$ with suitable simultaneous exchanges of two rows and two columns $(1\leftrightarrow k$ and $2\leftrightarrow l)$. These basis games can be considered as voluntary coordination games, because the neutral strategies (those with all-zero payoffs) can be interpreted as options to avoid participation in the coordination type interaction. If $\mathbf{A} = \mathbf{Q}^{(\mathrm{k,l})}$ then the game has two Nash equilibria when the players choose the same option, more precisely, the strategy pair $[k,k]$ or $[l,l]$. Opposite choices ($[k,l]$ or $[l,k]$) constitute Nash equilibria for $\mathbf{A} = -\mathbf{Q}^{(\mathrm{k,l})}$. Elementary coordination games resemble Ising-type interactions used widely in statistical physics [36–39]. Additionally, some combinations of these basis games lead to the Potts model [40] or the Ashkin-Teller model [41] when the players are located at the sites of a square lattice [42].

The antisymmetric matrix $\mathbf{A}^{(\mathrm{cyc})}$ can be built up from elementary basis matrices corresponding to a subset of "voluntary rock–paper–scissors games" for which the first strategy can always represent the choice of "rock". More precisely,

$$\mathbf{A}^{(\mathrm{cyc})} = \sum_{\substack{k,l\\1<k<l}} \lambda_{1kl}\mathbf{C}^{(1,\mathrm{k,l})}, \tag{8}$$

where

$$\mathbf{C}^{(1,2,3)} = \begin{pmatrix} 0 & +1 & -1 & 0 & \dots & 0 \\ -1 & 0 & +1 & 0 & \dots & 0 \\ +1 & -1 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & 0 & \dots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & 0 \end{pmatrix}, \tag{9}$$

and the other $\mathbf{C}^{(1,\mathrm{k,l})}$ independent elementary basis matrices can be obtained from $\mathbf{C}^{(1,2,3)}$ with simultaneous exchanges of two rows and two columns $(2\leftrightarrow k$ and $3\leftrightarrow l)$. On the one hand, this choice of elementary basis matrices is convenient, because it simplifies the determination of the coefficients: $\lambda_{1kl} = A^{(\mathrm{cyc})}_{kl}$ for $1<k<l$ while the remaining entries (those in the first row and column) are determined by the symmetry and orthogonality conditions. On the other hand, for this set of elementary basis matrices, the first strategy is distinguished, which calls the isomorphism of all voluntary rock–paper–scissors games into question. This discrepancy, however, is eliminated by the fact that the distinguished role of the first strategy can be transferred to any other strategy. The latter choice represents another set of elementary basis matrices.

In Eq. (8), the basis matrices $\mathbf{C}^{(1,\mathrm{k,l})}$ can each be considered as the adjacency matrix of a simple directed graph having n nodes and a directed three-edge loop connecting the nodes representing strategies 1, k, and l. Within the framework of graph theory, Eq. (8) expresses the simple topological feature that any directed loop can be built up from a suitable set of directed triangles [43]. Another interesting relation is that the antisymmetric (or zero-sum) part of $\mathbf{A}^{(\mathrm{se})}+\mathbf{A}^{(\mathrm{cr})}$ corresponds to a suitable sum of the adjacency matrices of starlike directed graphs. Thus, each symmetric zero-sum game can be represented by a simple directed weighted graph [44,45] that can be built up from starlike directed components and a suitable subset of directed triangles [46]. Furthermore, the orthogonality conditions allow us to measure the proportion of each orthogonal component by their projection onto $\mathbf{A}$. For example, the proportion of cyclic components [46] can be quantified as

$$\Theta = \frac{\mathbf{A}^{(\mathrm{cyc})}\cdot\mathbf{A}}{\mathbf{A}\cdot\mathbf{A}} = \frac{\mathbf{A}^{(\mathrm{cyc})}\cdot\mathbf{A}^{(\mathrm{cyc})}}{\mathbf{A}\cdot\mathbf{A}}, \tag{10}$$

where $0\le\Theta\le 1$. Evidently, similar quantities can measure the proportion of all other orthogonal components.

Cyclic elementary basis games describe situations in which selfish players are encouraged to modify their strategies cyclically in a chain of consecutive unilateral strategy changes. For example, if the interaction is defined by $\mathbf{C}^{(1,2,3)}$ then the active player can increase her payoff from -1 to $+1$ in each step along the loop $(1,3) \to (2,3) \to (2,1) \to (3,1) \to (3,2) \to (1,2) \to (1,3)$. Along these loops in the strategy space [47], the sum of payoff variations achieved by the active players ΔW is always positive. This type of interactions prevents the appearance of detailed balance when the forward and backward transitions occur with the same probability between two microscopic states.

Thermodynamical behavior (evolution towards the Boltzmann distribution or Gibbs ensemble, where detailed balance is satisfied) can be observed for potential games if the strategy updates are controlled by the logit rule, which is a noisy version of the best response dynamics [14,15,20,48,49]. The logit rule can be considered as a generalized version of the Glauber dynamics [50] used widely in statistical physics [16,51,52].

For potential games, one can derive a potential matrix $\mathbf{V}$, whose role is similar to that of the potential energy in physical systems with pair interactions between equivalent particles or objects with n states. The elements of $\mathbf{V}$ are defined by the relationships

$$V_{ij} - V_{kj} = A_{ij} - A_{kj} \text{ and } V_{ij} - V_{ik} = A_{ji} - A_{ki} \tag{11}$$

and they quantify the driving force for all unilateral strategy changes. Accordingly, $V_{ij} - V_{11}$ summarizes the increase of payoff obtained by the active player for consecutive unilateral strategy updates from the strategy pair $(1,1)$ to (i,j). The existence of the potential dictates that the variation of this potential should be independent of the path the players choose as the microscopic state is moved from (i,j) to (i',j') in the strategy space [47]. In other words, for potential games the sum of payoff variations received by the active players for all closed trajectories (loops) is zero. As detailed above, the latter condition is not satisfied for games containing cyclic ($\mathbf{C}^{(1,k,l)}$) components. Interestingly, the scalar product $\mathbf{A} \cdot \mathbf{C}^{(1,k,l)}$ is equivalent to the sum of payoff variations received by the active player for unilateral strategy updates along a four-edge loop $[(1,1) \to (k,1) \to (k,l) \to (1,l) \to (1,1)]$ in the strategy space. Thus, the potential can exist if the orthogonality criteria

$$\mathbf{A} \cdot \mathbf{C}^{(1,k,l)} = 0\,, \tag{12}$$

are satisfied for all $1 < k < l$. Here we have to emphasize that graph theoretical arguments [53] prove that all possible directed loops can be built up from directed triangles whose adjacency matrices are defined by the elementary cyclic components $\mathbf{C}^{(1,k,l)}$ in Eq. (8). Consequently, the above orthogonality conditions are sufficient to justify the existence of the potential.

Eq. (11) ensures that the potential matrix can be evaluated separately for each type of games if $\mathbf{A}^{(cyc)} = 0$ and it simplifies the determination of the potential matrix, because $\mathbf{V} = \mathbf{A}$ if $\mathbf{A}$ is symmetric, that is, if $\mathbf{A} = \mathbf{A}^{+}$. The self-dependent component also provides a symmetric contribution, which means that $\mathbf{V} = \mathbf{A}^{(se)} + \mathbf{A}^{(se)+}$ if $\mathbf{A} = \mathbf{A}^{(se)}$. Furthermore, the components $\mathbf{A}^{(cr)}$ and $a\mathbf{U}$ both give an irrelevant (zero or constant) contribution to $\mathbf{V}$. In sum, the potential matrix exists if $\mathbf{A}^{(cyc)} = 0$, it is symmetric, and it obeys a simple form:

$$\mathbf{V} = c\mathbf{U} + \mathbf{A}^{(se)} + \mathbf{A}^{(se)+} + \mathbf{A}^{(co)}\,, \tag{13}$$

where that value of c can be chosen arbitrarily.

Potential games have some peculiarities. For example, the (single) largest value of the matrix elements V_{ij} defines a strict Nash equilibrium. If pair interactions are defined by a potential game in an evolutionary game in which the strategy updates (the dynamics) is controlled by the logit rule then the system evolves into a Gibbs ensemble [14,54]. Consequently, the macroscopic behavior in the stationary states of these systems can be evaluated by the methods of statistical physics [16,51,52,55]. In the corresponding spatial evolutionary games the stationary states are not affected by the component $\mathbf{A}^{(cr)}$ but the average payoffs involve its contributions [56,57]. Due to this feature, social dilemmas can occur in potential games, too. Furthermore, the absence of real pair interactions in purely cross-dependent ($\mathbf{G} = \mathbf{A}^{(cr)}$) games means that the macroscopic behavior of such systems can be described by considering just a single player.

Additionally, we have to mention that games of the form $\mathbf{A}^{(se)} + \mathbf{A}^{(cr)}$ can be considered as a generalization of the donation game [35] with n strategies. The symmetric part of $\mathbf{A}^{(se)} + \mathbf{A}^{(cr)}$ is identical to a term describing the effect of an external field on every individual state in many-particle systems. In these systems, the individual behavior of players is not affected by the others, and thus the thermodynamical state can be determined by considering only a single player.

In contrast, the antisymmetric part of $\mathbf{A}^{(se)} + \mathbf{A}^{(cr)}$ is responsible for the emergence of social dilemmas, when selfish behavior may dictate the choice of a Nash equilibrium that is not optimal for the players [16,34,35,58]. It is worth noting, furthermore, that antisymmetric payoff matrices describe zero-sum games (with payoff pairs A_{ij} and $-A_{ij}$), while symmetric payoff matrices define partnership or fraternal games, in which the players receive equivalent incomes [16,22,59].

3. Decomposition of bimatrix games

Let us now consider bimatrix games with two distinct players (called Alice and Bob) who each have n strategies and their payoffs are defined by two payoff matrices, $\mathbf{A}$ and $\mathbf{B}$. More precisely, the payoffs for Alice and Bob are A_{ij} and B_{ij} for the choice of strategy pair $[i,j]$. Using the standard bimatrix formalism, these games can be denoted as

$$\mathbf{G} = \begin{pmatrix} (A_{11}, B_{11}) & (A_{12}, B_{12}) & \dots & (A_{1n}, B_{1n}) \\ (A_{21}, B_{21}) & (A_{22}, B_{22}) & \dots & (A_{2n}, B_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (A_{n1}, B_{n1}) & (A_{n2}, B_{n2}) & \dots & (A_{nn}, B_{nn}) \end{pmatrix}. \tag{14}$$

A bimatrix game is a symmetric game if $\mathbf{B} = \mathbf{A}^{+}$.

Similarly to symmetric games, bimatrix games can also be separated into the sum of orthogonal components. In this formalism, the orthogonality between two games ($\mathbf{G}$ and $\tilde{\mathbf{G}}$) given by pairs of payoff matrices [($\mathbf{A}, \mathbf{B}$) and ($\tilde{\mathbf{A}}, \tilde{\mathbf{B}}$)] is defined via the generalization of the matrix scalar product as

$$\mathbf{G} \cdot \tilde{\mathbf{G}} = \sum_{i,j} A_{ij} \tilde{A}_{ij} + \sum_{i,j} B_{ij} \tilde{B}_{ij} = 0 . \tag{15}$$

In contrast to symmetric games, here we distinguish two orthogonal games ($\mathbf{G}^{(\mathrm{avA})}$ and $\mathbf{G}^{(\mathrm{avB})}$ with $\mathbf{G}^{(\mathrm{avA})} \cdot \mathbf{G}^{(\mathrm{avB})} = 0$) characterizing irrelevant constants,

$$\mathbf{G}^{(\mathrm{avA})} = a \begin{pmatrix} (1,0) & (1,0) & \dots & (1,0) \\ (1,0) & (1,0) & \dots & (1,0) \\ \vdots & \vdots & \ddots & \vdots \\ (1,0) & (1,0) & \dots & (1,0) \end{pmatrix} \text{ and } \mathbf{G}^{(\mathrm{avB})} = b \begin{pmatrix} (0,1) & (0,1) & \dots & (0,1) \\ (0,1) & (0,1) & \dots & (0,1) \\ \vdots & \vdots & \ddots & \vdots \\ (0,1) & (0,1) & \dots & (0,1) \end{pmatrix} , \tag{16}$$

where

$$a = \frac{1}{n^2} \sum_{i,j} A_{ij} \text{ and } b = \frac{1}{n^2} \sum_{i,j} B_{ij} \tag{17}$$

are the average values of the payoffs of Alice and Bob.

We also define the self-dependent orthogonal components separately for the two players as

$$\mathbf{G}^{(\mathrm{seA})} = \begin{pmatrix} (\varepsilon_1,0) & (\varepsilon_1,0) & \dots & (\varepsilon_1,0) \\ (\varepsilon_2,0) & (\varepsilon_2,0) & \dots & (\varepsilon_2,0) \\ \vdots & \vdots & \ddots & \vdots \\ (\varepsilon_n,0) & (\varepsilon_n,0) & \dots & (\varepsilon_n,0) \end{pmatrix} \text{ and } \mathbf{G}^{(\mathrm{seB})} = \begin{pmatrix} (0,\epsilon_1) & (0,\epsilon_2) & \dots & (0,\epsilon_n) \\ (0,\epsilon_1) & (0,\epsilon_2) & \dots & (0,\epsilon_n) \\ \vdots & \vdots & \ddots & \vdots \\ (0,\epsilon_1) & (0,\epsilon_2) & \dots & (0,\epsilon_n) \end{pmatrix} , \tag{18}$$

where

$$\varepsilon_i = \frac{1}{n} \sum_j A_{ij} - a \text{ and } \epsilon_j = \frac{1}{n} \sum_i B_{ij} - b . \tag{19}$$

These definitions of ε_i and ϵ_j ensure orthogonality to the average payoff components, that is, $\mathbf{G}^{(\mathrm{seA})} \cdot \mathbf{G}^{(\mathrm{seB})} = \mathbf{G}^{(\mathrm{seA})} \cdot \mathbf{G}^{(\mathrm{avA})} = \mathbf{G}^{(\mathrm{seA})} \cdot \mathbf{G}^{(\mathrm{avB})} = \mathbf{G}^{(\mathrm{seB})} \cdot \mathbf{G}^{(\mathrm{avA})} = \mathbf{G}^{(\mathrm{seB})} \cdot \mathbf{G}^{(\mathrm{avB})} = 0$, because

$$\sum_i \varepsilon_i = 0 \text{ and } \sum_j \epsilon_j = 0 . \tag{20}$$

We introduce the cross-dependent components similarly too, that is, as

$$\mathbf{G}^{(\mathrm{crA})} = \begin{pmatrix} (\gamma_1,0) & (\gamma_2,0) & \dots & (\gamma_n,0) \\ (\gamma_1,0) & (\gamma_2,0) & \dots & (\gamma_n,0) \\ \vdots & \vdots & \ddots & \vdots \\ (\gamma_1,0) & (\gamma_2,0) & \dots & (\gamma_n,0) \end{pmatrix} \text{ and } \mathbf{G}^{(\mathrm{crB})} = \begin{pmatrix} (0,\delta_1) & (0,\delta_1) & \dots & (0,\delta_1) \\ (0,\delta_2) & (0,\delta_2) & \dots & (0,\delta_2) \\ \vdots & \vdots & \ddots & \vdots \\ (0,\delta_n) & (0,\delta_n) & \dots & (0,\delta_n) \end{pmatrix} , \tag{21}$$

where the entries are defined by and

$$\gamma_j = \frac{1}{n} \sum_i A_{ij} - a \text{ and } \delta_i = \frac{1}{n} \sum_j B_{ij} - b . \tag{22}$$

As a result,

$$\sum_j \gamma_j = 0 \text{ and } \sum_i \delta_i = 0, \tag{23}$$

which again ensure orthogonality to both the average and the self-dependent components, meaning $\mathbf{G}^{(\mathrm{crA})} \cdot \mathbf{G}^{(\mathrm{crB})} = \mathbf{G}^{(\mathrm{crA})} \cdot \mathbf{G}^{(\mathrm{avA})} = \mathbf{G}^{(\mathrm{crA})} \cdot \mathbf{G}^{(\mathrm{avB})} = \mathbf{G}^{(\mathrm{crB})} \cdot \mathbf{G}^{(\mathrm{avA})} = \mathbf{G}^{(\mathrm{crB})} \cdot \mathbf{G}^{(\mathrm{avB})} = 0$.

The construction of the orthogonal components mentioned above is closely related to the methods that proved to be fruitful for symmetric games. The $\mathbf{G}' = \mathbf{G} - \mathbf{G}^{(\mathrm{avA})} - \mathbf{G}^{(\mathrm{avB})} - \mathbf{G}^{(\mathrm{seA})} - \mathbf{G}^{(\mathrm{seB})} - \mathbf{G}^{(\mathrm{crA})} - \mathbf{G}^{(\mathrm{crB})}$ remainder of the bimatrix consists of payoff matrices $\mathbf{A}'$ and $\mathbf{B}'$ in which the sum of the entries is zero in each row and column. As an aside worth mentioning, having this property ensures that both players receive zero average payoff if the other player picks their strategy uniformly randomly, which means that both players choosing their strategy this way is a non-strict Nash equilibrium [24]. Such games can also be separated into the sum of a so-called fraternal game part and a zero-sum game part as

$$\mathbf{G}' = \mathbf{G}^{(\mathrm{f})} + \mathbf{G}^{(\mathrm{z})} . \tag{24}$$

The fraternal component takes the form

$$\mathbf{G}^{(\mathrm{f})} = \begin{pmatrix} (F_{11},F_{11}) & (F_{12},F_{12}) & \dots & (F_{1n},F_{1n}) \\ (F_{21},F_{21}) & (F_{22},F_{22}) & \dots & (F_{2n},F_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (F_{n1},F_{n1}) & (F_{n2},F_{n2}) & \dots & (F_{nn},F_{nn}) \end{pmatrix} , \tag{25}$$

where $\mathbf{F} = (\mathbf{A}' + \mathbf{B}')/2$. The zero-sum component of a bimatrix game obeys the form

$$\mathbf{G}^{(z)} = \begin{pmatrix} (E_{11}, -E_{11}) & (E_{12}, -E_{12}) & \dots & (E_{1n}, -E_{1n}) \\ (E_{21}, -E_{21}) & (E_{22}, -E_{22}) & \dots & (E_{2n}, -E_{2n}) \\ \vdots & \vdots & \ddots & \vdots \\ (E_{n1}, -E_{n1}) & (E_{n2}, -E_{n2}) & \dots & (E_{nn}, -E_{nn}) \end{pmatrix}, \tag{26}$$

with $\mathbf{E} = (\mathbf{A}' - \mathbf{B}')/2$. The $\mathbf{F}$ and $\mathbf{E}$ matrices inherit the general features of $\mathbf{A}'$ and $\mathbf{B}'$, yielding

$$\sum_i F_{ij} = \sum_j F_{ij} = \sum_i E_{ij} = \sum_j E_{ij} = 0. \tag{27}$$

The decomposition of $\mathbf{G}^{(f)}$ into elementary components is simplified by further separating the matrix $\mathbf{F}$ into the sum of its symmetric and antisymmetric parts. The contribution of the symmetric part of $\mathbf{F}$ [i.e., $(\mathbf{F}+\mathbf{F}^{+})/2$] coincides with the coordination-type interactions discussed in the previous section. Using the bimatrix formalism, this term can be written as

$$\mathbf{G}^{(co)} = \begin{pmatrix} (v_{11}, v_{11}) & (-v_{12}, -v_{12}) & (-v_{13}, -v_{13}) & \dots & (-v_{1n}, -v_{1n}) \\ (-v_{12}, -v_{12}) & (v_{22}, v_{22}) & (-v_{23}, -v_{23}) & \dots & (-v_{2n}, -v_{2n}) \\ (-v_{13}, -v_{13}) & (-v_{23}, -v_{23}) & (v_{33}, v_{33}) & \dots & (-v_{3n}, -v_{3n}) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (-v_{1n}, -v_{1n}) & (-v_{2n}, -v_{2n}) & (-v_{3n}, -v_{3n}) & \dots & (v_{nn}, v_{nn}) \end{pmatrix}, \tag{28}$$

where the symmetric non-diagonal elements v_{ij} $(i < j)$ measure the strength of coordination between the strategy pair $[i, j]$ (the same as for symmetric games) and the diagonal elements are determined by the orthogonality conditions [see Eq. (27)].

The antisymmetric part of $\mathbf{F}$ [i.e., $(\mathbf{F}-\mathbf{F}^{+})/2$] defines a new type of interaction, which can also be built up from a set of isomorphic elementary basis games. In the bimatrix formalism, one of these basis games can be given as

$$\mathbf{D}^{(1,2,3)} = \begin{pmatrix} (0,0) & (1,1) & (-1,-1) & (0,0) & \dots & (0,0) \\ (-1,-1) & (0,0) & (1,1) & (0,0) & \dots & (0,0) \\ (1,1) & (-1,-1) & (0,0) & (0,0) & \dots & (0,0) \\ (0,0) & (0,0) & (0,0) & (0,0) & \dots & (0,0) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (0,0) & (0,0) & (0,0) & (0,0) & \dots & (0,0) \end{pmatrix}. \tag{29}$$

The main features of this game are: (1) $\mathbf{D}^{(1,2,3)}$ is orthogonal to the elementary types discussed above; (2) the total benefit is shared equally by Alice and Bob; (3) it can be considered as an anticoordination-type interaction favoring three strategy pairs ($[1, 2]$, $[2, 3]$, and $[3, 1]$) equally while exchanging these strategies turns the total benefit into a loss; (4) the players' income is zero if they choose identical strategies; (5) the neutral strategies (here $i, j > 3$) represent voluntarism, providing different ways to avoid participation. Due to these features, $\mathbf{D}^{(1,2,3)}$ can be interpreted as a voluntary directed anticoordination game that quantifies the mutual advantage or disadvantage the players derive from the difference in their capabilities (or skills) when choosing different strategy pairs. Evidently, similar interactions can be introduced involving any three distinguished strategies. For example, the elementary basis game $\mathbf{D}^{(i,j,k)}$ can be derived from $\mathbf{D}^{(1,2,3)}$ via simultaneously exchanging three suitable pairs of lines and columns in Eq. (29), namely, $1 \leftrightarrow i$, $2 \leftrightarrow j$, and $3 \leftrightarrow k$, if $i < j < k$. The similarity between $\mathbf{C}^{(1,2,3)}$ and $\mathbf{D}^{(1,2,3)}$ implies that the whole set spanned by elementary directed anticoordination bimatrix games can be given similarly to symmetric cyclic dominance games, that is, as

$$\mathbf{G}^{(dco)} = \sum_{\substack{k,l \\ 1<k<l}} \Delta_{1kl} \mathbf{D}^{(1,k,l)}, \tag{30}$$

where the coefficients define the equal payoffs (Δ_{1kl} and $-\Delta_{1kl}$) of Alice and Bob when they choose the strategy pairs $[k, l]$ and $[l, k]$ $[1 < k < l]$, respectively, in the game $\mathbf{G}^{(dco)}$. For this set of elementary basis games, the players' income is prescribed by the orthogonality conditions [see Eq. (27)] if one of them chooses the first strategy. Similarly to symmetric games, the distinguished role of the first strategy can also be assumed by any other strategy.

When following the previous approach, the zero-sum component $\mathbf{G}^{(z)}$ is divided into two parts by separating the symmetric and antisymmetric parts of $\mathbf{E}$ as $\mathbf{G}^{(z)} = \mathbf{G}^{(mp)} + \mathbf{G}^{(as)}$. The $\mathbf{G}^{(as)}$ component given by the antisymmetric part of $\mathbf{E}$ [i.e., $(\mathbf{E} - \mathbf{E}^{+})/2$] is equivalent to the cyclic component of symmetric games. Thus, it can be built up as a linear combination of voluntary rock–paper–scissors games in the same way as discussed in the previous section [see Eq. (8)]. If $i = 1$ is chosen to be the distinguished strategy then $\mathbf{G}^{(as)}$ can be given as

$$\mathbf{G}^{(as)} = \begin{pmatrix} (0,0) & (\lambda_{12}, -\lambda_{12}) & (\lambda_{13}, -\lambda_{13}) & (\lambda_{14}, -\lambda_{14}) & \dots & (\lambda_{1n}, -\lambda_{1n}) \\ (-\lambda_{12}, \lambda_{12}) & (0,0) & (\lambda_{123}, -\lambda_{123}) & (\lambda_{124}, -\lambda_{124}) & \dots & (\lambda_{12n}, -\lambda_{12n}) \\ (-\lambda_{13}, \lambda_{13}) & (-\lambda_{123}, \lambda_{123}) & (0,0) & (\lambda_{134}, -\lambda_{134}) & \dots & (\lambda_{13n}, -\lambda_{13n}) \\ (-\lambda_{14}, \lambda_{14}) & (-\lambda_{124}, \lambda_{124}) & (-\lambda_{134}, \lambda_{134}) & (0,0) & \dots & (\lambda_{14n}, -\lambda_{14n}) \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ (-\lambda_{1n}, \lambda_{1n}) & (-\lambda_{12n}, \lambda_{12n}) & (-\lambda_{13n}, \lambda_{13n}) & (-\lambda_{14n}, \lambda_{14n}) & \dots & (0,0) \end{pmatrix}, \tag{31}$$

where λ_{1kl} $(1 < k < l)$ defines the strength of the elementary cyclic dominance involving strategy triple $(1, k, l)$ and the values of λ_{1k} are prescribed by the orthogonality conditions [see Eq. (27)]. Evidently, this type of component is missing from $n = 2$ bimatrix games. In these latter games, the existence of a potential can only be prevented by the presence of a matching pennies interaction component. The two-strategy matching pennies game can also be extended by introducing neutral strategies, that is, adding rows and columns with zero entries. In the corresponding zero-sum games, $\mathbf{M}^{(\mathrm{i,j|k,l})}$, Alice wins when the players choose strategy pairs $[i,k]$ and $[j,l]$, she loses when the choices are $[j,k]$ and $[i,l]$, and neither player receives any payoff otherwise. Using the formalism of bimatrix games, a symmetric version ($i=k$, $j=l$, and $i<j$) of this voluntary matching pennies game obeys the form:

$$\mathbf{M}^{(1,2|1,2)}=\begin{pmatrix} (1,-1) & (-1,1) & (0,0) & \dots & (0,0) \\ (-1,1) & (1,-1) & (0,0) & \dots & (0,0) \\ (0,0) & (0,0) & (0,0) & \dots & (0,0) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ (0,0) & (0,0) & (0,0) & \dots & (0,0) \end{pmatrix}, \tag{32}$$

and the other symmetric basis games $\mathbf{M}^{(\mathrm{i,j|i,j})}$ can be obtained by simultaneously exchanging the same two rows and two columns of $\mathbf{M}^{(1,2|1,2)}$ ($1\leftrightarrow i$ and $2\leftrightarrow j$). Note that all $\mathbf{M}^{(\mathrm{i,j|i,j})}$ are orthogonal to the previously discussed components, which guarantees that the linear combinations of this subset of basis games span a subspace of the parameter space that is linearly independent from those discussed above. Thus, similarly to symmetric coordination games, $\mathbf{G}^{(\mathrm{mp})}$ can be given through $n(n-1)/2$ parameters:

$$\mathbf{G}^{(\mathrm{mp})}=\sum_{k<l}\Lambda_{kl}\mathbf{M}^{(\mathrm{k,l|k,l})}. \tag{33}$$

Non-symmetric voluntary matching pennies games ($\mathbf{M}^{(\mathrm{i,j|k,l})}$ with $i\neq k$ and/or $j\neq l$) are related to voluntary rock–paper–scissors basis games [53] in a way illustrated for $n=3$ by

$$\begin{pmatrix} (0,0) & (1,-1) & (-1,1) \\ (-1,1) & (0,0) & (1,-1) \\ (1,-1) & (-1,1) & (0,0) \end{pmatrix}=\begin{pmatrix} (0,0) & (1,-1) & (-1,1) \\ (0,0) & (-1,1) & (1,-1) \\ (0,0) & (0,0) & (0,0) \end{pmatrix}-\begin{pmatrix} (0,0) & (0,0) & (0,0) \\ (1,-1) & (-1,1) & (0,0) \\ (-1,1) & (1,-1) & (0,0) \end{pmatrix}. \tag{34}$$

Additionally, the $\mathbf{M}^{(\mathrm{i,j|k,l})}$ games are not independent. It is easy to check that,

$$\mathbf{M}^{(\mathrm{i,j|i,j+2})}=\mathbf{M}^{(\mathrm{i,j|i,j+1})}+\mathbf{M}^{(\mathrm{i,j+1|i,j+2})}, \tag{35}$$

if $j+2\leq n$. Similar relations allow us to use other complete and independent sets of $\mathbf{M}^{(\mathrm{i,j|k,l})}$ games to express $\mathbf{G}^{(\mathrm{z})}$. For example,

$$\mathbf{G}^{(\mathrm{z})}=\sum_{1<i,j}\alpha_{ij}\mathbf{M}^{(\mathrm{1,i|1,j})}\ \text{ or }\ \mathbf{G}^{(\mathrm{z})}=\sum_{i,j<(n-1)}\beta_{ij}\mathbf{M}^{(\mathrm{i,j|i+1,j+1})}. \tag{36}$$

In both formulae, $(n-1)^2$ coefficients define this type of interaction. Graph theoretical arguments in support of these formulae are given in Ref. [53].

All fraternal games admit a potential, because the sum of the individual payoff variations (for the active player) is zero along all rectangular four-edge loops in the space of strategy pairs (e.g., $[i,j]\to[i,j']\to[i',j']\to[i',j]\to[i,j]$). The elements of the corresponding potential matrix are equivalent to the payoffs received by the players. For example, $\mathbf{V}=\mathbf{F}$ [henceforth, the presence of the irrelevant constant, $c\mathbf{U}$, is omitted for the sake of simplicity] if the game is of the form prescribed by Eq. (25). We emphasize that this potential matrix itself is not symmetric in the presence of directed anticoordination terms [given by Eq. (30)].

As mentioned above, no potential exists in the presence of matching pennies-type elementary interactions. Furthermore, the contributions of $\mathbf{G}^{(\mathrm{avA})}$ and $\mathbf{G}^{(\mathrm{avB})}$ are irrelevant with respect to the potential. Similarly to symmetric games, the cross-dependent components also give zero contributions to $\mathbf{V}$. Thus, the potential matrix obeys a simple form if $\mathbf{G}=\mathbf{G}^{(\mathrm{seA})}+\mathbf{G}^{(\mathrm{seB})}$:

$$\mathbf{V}^{(\mathrm{se})}=\begin{pmatrix} (\varepsilon_1+\epsilon_1) & (\varepsilon_1+\epsilon_2) & \dots & (\varepsilon_1+\epsilon_n) \\ (\varepsilon_2+\epsilon_1) & (\varepsilon_2+\epsilon_2) & \dots & (\varepsilon_2+\epsilon_n) \\ \vdots & \vdots & \ddots & \vdots \\ (\varepsilon_n+\epsilon_1) & (\varepsilon_n+\epsilon_2) & \dots & (\varepsilon_n+\epsilon_n) \end{pmatrix}, \tag{37}$$

which is defined by $2(n-1)$ independent parameters. $\mathbf{V}^{(\mathrm{se})}$ quantifies external preferences similar to the application of an external field in Ising-type models. Evidently, this term may favor different choices for Alice and Bob. Note that, unlike for symmetric games, the contribution of the self-dependent component to the potential matrix of a general bimatrix game can have a non-zero antisymmetric part.

Social dilemmas may occur in the presence of $\mathbf{G}^{(\mathrm{cr})}=\mathbf{G}^{(\mathrm{crA})}+\mathbf{G}^{(\mathrm{crB})}$, which modifies payoffs without affecting the incentives of selfish players, possibly reducing the players' incomes in the preferred Nash equilibrium. This feature significantly expands the spectrum of social dilemmas, which may include interaction situations yet to be investigated.

4. Spatial example with directed anticoordination interactions

Multiplayer bimatrix games with nearest-neighbor interactions can only be readily set up on bipartite networks or lattices consisting of two non-overlapping sets of sites, which accommodate neighboring players being of different types (a and b). The best-known example is the chessboardlike arrangement of players on a square lattice. On these connectivity structures, the stationary spatial distribution of strategies can exhibit order–disorder phase transitions as the noise level of the logit rule governing the strategy choices is varied when the pair interaction is of the form $\mathbf{G}^{(\mathrm{co})}+\mathbf{G}^{(\mathrm{dco})}$ [16]. The universal features of such phase transitions are investigated exhaustively in the literature of statistical physics [29,60], especially if $\mathbf{G}=\mathbf{G}^{(\mathrm{co})}$ and the corresponding $\mathbf{V}$ is Hermitian.

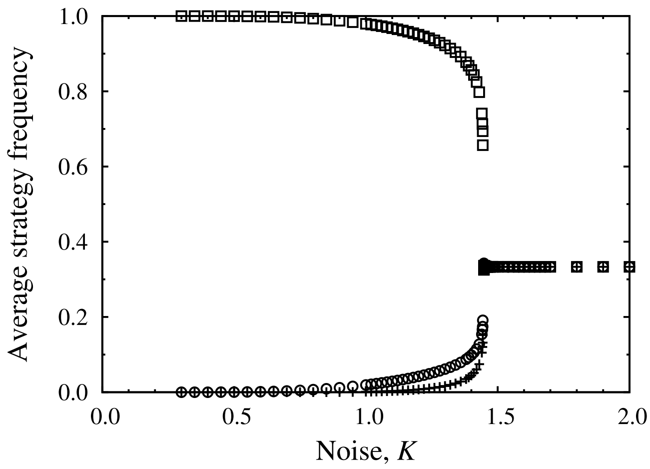

Fig. 1. The noise-dependence of average strategy frequencies (boxes: $\langle\rho_{a,1}\rangle = \langle\rho_{b,2}\rangle$, open circles: $\langle\rho_{a,2}\rangle = \langle\rho_{b,3}\rangle$, and pluses: $\langle\rho_{a,3}\rangle = \langle\rho_{b,1}\rangle$) on the chessboardlike sublattices show an order–disorder phase transition: Above the critical temperature, all strategies are present in equal frequency on both sublattices; whereas below the critical temperature, symmetry is spontaneously broken, and the frequencies of strategy 1 on sublattice a and strategy 2 on sublattice b increase equally as $K \to 0$, while strategy pairs 2–3 and 3–1 decrease in a similarly anticoordinated manner.

Non-Hermitian interactions [61] involving $\mathbf{G}^{(\mathrm{dco})}$ components, however, may give rise to new universal behaviors yet to be explored. Additional novelties may emerge due to further non-Hermitian contributions of the antisymmetric part of $\mathbf{G}^{(\mathrm{se})}$, which represents staggered external fields [62–64] in these spatial models.

Now we study a three-strategy evolutionary game with the chessboardlike arrangement of players on a square lattice when the pair interactions between the nearest neighbors are defined by a uniform directed anticoordination game $\mathbf{D}^{(1,2,3)}$ for $n = 3$ [see Eq. (29)]. For this two-player interaction the non-Hermitian pair potential can be defined by an antisymmetric matrix, that is,

$$\mathbf{V} = \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}. \tag{38}$$

It is noteworthy that the main features of this interaction remain unchanged when $-\mathbf{D}^{(1,2,3)}$ defines the interaction because this modification is equivalent to exchanging two strategy labels for both players (e.g., $1 \leftrightarrow 2$). A similar isomorphism is valid for the traditional rock–paper–scissors games.

For the sake of direct comparability with the Hermitian-potential models of statistical physics, we choose the evolution of strategy distribution to be controlled by the logit rule [14,16,54], which favors exponentially the higher payoff for unilateral strategy updates for players selected at random. More precisely, the player at site x chooses the strategy $\mathbf{s}_x$ with a probability

$$w(\mathbf{s}_x) = \frac{e^{u_x(\mathbf{s}_x)/K}}{\sum_{\mathbf{s}'_x} e^{u_x(\mathbf{s}'_x)/K}} \tag{39}$$

where $u_x(\mathbf{s}_x)$ is the accumulated payoff from games with the four nearest neighbors and K quantifies the noise level (similar to temperature in physical systems). In the stationary state the K-dependent macroscopic behavior is quantified by the average strategy frequencies $\langle\rho_{z,i}\rangle$ for $i = 1, 2, 3$ on the sublattices $z = a$ or b. Evidently, $\sum_i \langle\rho_{z,i}\rangle = 1$.

Using traditional Monte Carlo (MC) [30] simulations we have determined the K-dependence of the average strategy frequencies on both sublattices containing $N = L^2/2$ sites (L is the linear size of the system). Similarly to the three-state Potts model [40] this evolutionary game has three equivalent ordered strategy arrangements, namely, $\langle\rho_{a,1}\rangle = \langle\rho_{b,2}\rangle = 1$, $\langle\rho_{a,2}\rangle = \langle\rho_{b,3}\rangle = 1$, and $\langle\rho_{a,3}\rangle = \langle\rho_{b,1}\rangle = 1$, which may occur with equivalent probabilities in the limit $K \to 0$, if the MC simulation is started from a random initial state. Fig. 1 illustrates a critical order–disorder phase transition at $K = K_c = 1.4444(1)$ when K is increased.

Most of our MC data were obtained on a square lattice with a linear size $L = 1000$ and averaging over a sampling time $t_s = 10^5$ MCS after a relaxation time $t_r = 10^5$ MCS. In the close vicinity of the critical point, however, we had to use larger sizes ($L \leq 2000$) and longer times (t_s and t_r) to reduce the statistical uncertainties required for the identification of the universality class of this critical phase transition.

The three strategies appear with equivalent frequencies (1/3) on both sublattices when $K > K_c$. More precisely, on the chessboardlike sublattices the first ordered phase can be described by two order parameters (m_1 and m_2), that is, $\langle\rho_{a,1}\rangle = \langle\rho_{b,2}\rangle = 1/3 + m_1 + m_2$, $\langle\rho_{a,2}\rangle = \langle\rho_{b,3}\rangle = 1/3 - m_1$, and $\langle\rho_{a,3}\rangle = \langle\rho_{b,1}\rangle = 1/3 - m_2$ with $0 \leq m_1, m_2 \leq 1/3$. The average strategy frequencies for the other two ordered phases can be obtained from the previous one with cyclic permutation of strategy indices.

In agreement with expectations, the order parameters exhibit power-law behavior (see Fig. 2) in the vicinity of the critical point, that is, $m_1, m_2 \propto (K - K_c)^{\beta}$ with an exponent close to $\beta = 1/9$ characteristic to the universality class of the three-state Potts model [16,40,65].

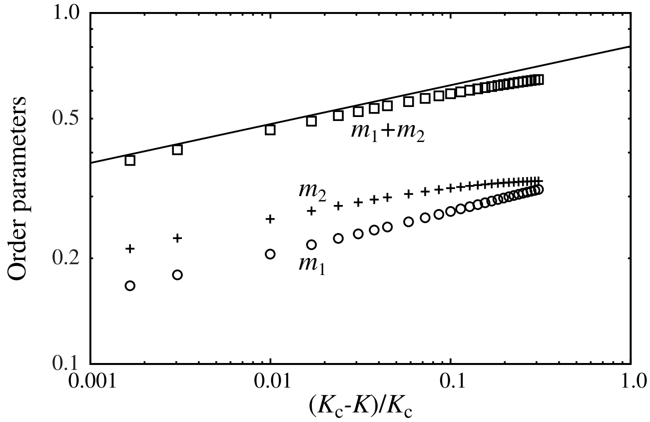

Fig. 2. Power law behavior of order parameters when $K \to K_c$ from below. The solid line (with slope 1/9) illustrates the universal behavior for the two-dimensional three-state Potts model, which seems to fit the MC data well, suggesting that the phase transition of the directed anticoordination game belongs to the same universality class. Symbols are the same as in Fig. 1.

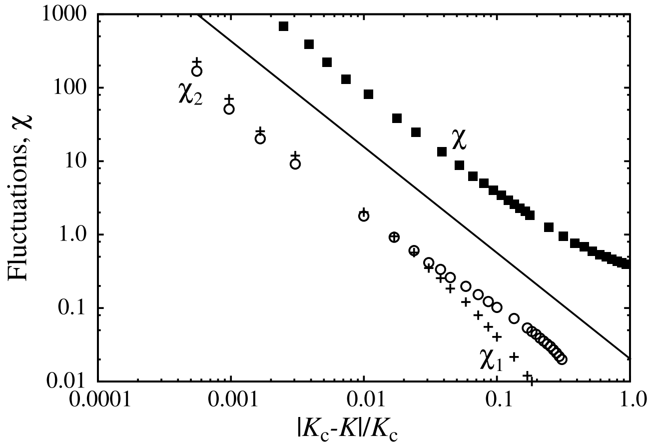

Fig. 3. The fluctuations of the strategy frequencies show universal power-law behavior if $K \to K_c$. Filled boxes show the average fluctuations (χ) of the three strategies on both sublattices if $K \to K_c$ from above. Open circles and pluses (similarly to Figs. 1 and 2) show χ_1 and χ_2 below the critical point. The solid line with slope $-13/9$ illustrates the divergency for the two-dimensional three-state Potts model, further indicating that the directed anticoordination game belongs to the same universality class.

It is noteworthy that for the present model the ordered phase is described by two parameters (m_1 and m_2) while for the Potts model the ordered phase can be characterized by a single (one-dimensional) order parameter corresponding to $m_1 = m_2$. This difference between the models has induced us to consider other quantities diverging in the vicinity of the critical point. Thus, we have also determined the fluctuations, quantified as,

$$\chi_1 = N[\langle \rho_{a,2}^2 \rangle - (1/3 - m_1)^2], \tag{40}$$

$$\chi_2 = N[\langle \rho_{a,3}^2 \rangle - (1/3 - m_2)^2], \tag{41}$$

if $K < K_c$ and

$$\chi = \frac{N}{6} \sum_{z,i} [\langle \rho_{z,i}^2 \rangle - 1/9], \tag{42}$$

if $K > K_c$. The application of the last formula reduces the statistical errors.

The MC results (see Fig. 3) support that this transition belongs to the universality class represented by the two-dimensional three-state Potts model predicting power law divergency (e.g., $\chi \propto |K - K_c|^{-\gamma}$) in the fluctuations with an exponent $\gamma = 13/9$ on both sides of the critical point [16,60,66].

Finally, it is worth mentioning that the values of K_c and γ agree within statistical error. The clarification of this agreement as well as the systematic consideration of other effects (e.g., external fields) represent challenges for later research of systems with non-reciprocal interactions [31,67,68].

5. Summary

Matrices and the games defined by them, just like vectors, can be built up as a linear combination of orthogonal (or at least independent) basis matrices. In the present work, we adapted the concept of matrix decomposition to the classification of interactions in n-strategy bimatrix games. The results show similarities and relevant differences between symmetric and bimatrix games. In both cases, this analysis reveals four fundamentally different types of interaction in addition to an irrelevant constant term, which is doubled for bimatrix games. These four orthogonal types of interaction possess well-distinguished symmetries (e.g., equivalent payoffs in the rows and columns of the payoff matrices) or conditions (the sum of payoffs is zero in every row and column) that remain unchanged when the strategy labels are permutated. This feature is consistent with the concept of isomorphism and leaves the type of interaction unchanged when strategy labels are exchanged. The isomorphism reduces the range of possible different interactions, while matrix decomposition provides a simple mathematical frame for the systematic exploration of the relationships between the interaction and resultant behavior in multiagent evolutionary games. From the viewpoint of practice, this mathematical frame supports the identification of isomorphic games as well as the explanation of similar behavior observed in systems developed for consideration of phenomena in fundamentally different fields of science.

For bimatrix games, the number of free parameters (dimensions) that determine the self- and cross-dependent payoff components of a game ($\mathbf{G}^{(\mathrm{seA})} + \mathbf{G}^{(\mathrm{seB})} + \mathbf{G}^{(\mathrm{crA})} + \mathbf{G}^{(\mathrm{crB})}$) are also doubled with respect to symmetric games. These terms quantify external effects, which affect winnings based on the strategy choice of just one of the players instead of the outcome of a proper player–player interaction. Games that combine only self- and cross-dependent payoffs can be considered as a generalization of donation games with multiple strategy-dependent costs and benefits. This enhanced freedom may yield new types of social dilemma situations, in which the individual incentives of selfish players lead to suboptimal outcomes that fall short of the highest achievable payoffs. Such features could even persist in the presence of further components, like in prisoner's dilemma or hawk–dove games – both $n = 2$ symmetric games –, which also involve coordination-type interactions.

The expansion coefficients (or strength parameters) of the irrelevant constant and self- and cross-dependent payoff terms can be easily identified for both symmetric and bimatrix games. The remaining parts of games can be separated into a fraternal ($\mathbf{G}^{(\mathrm{f})}$) and a zero-sum ($\mathbf{G}^{(\mathrm{z})}$) component. In fraternal (or partnership) games, the players receive equal payoffs, so their individual and common interests coincide, similarly to particle systems, where the variation of total energy drives evolution. The fraternal component of symmetric games can be built up from traditional coordination-type (Ising-type) interactions between each strategy pair. Applied to bimatrix games, this approach throws light on so-called directed anticoordination components, which describe the advantage or disadvantage of anticoordination between players with complementing skills or competence. In other words, the directed anticoordination game quantifies the driving force of specialization, labor division, and diversification in living systems.

In the present approach, the separation of a matrix into its symmetric and antisymmetric parts also plays a crucial role when the remainder of the zero-sum part of a game ($\mathbf{G}^{(\mathrm{z})}$) is built up as a linear combination of symmetric voluntary matching pennies games and voluntary rock–paper–scissors games. The isomorphism becomes more visible when the voluntary rock–paper–scissors components are replaced by a suitable set of non-symmetric voluntary matching pennies games. Despite the large number [$n^2(n-1)^2/4$] of possible voluntary matching pennies games, the contribution of these interactions can be quantified using just $(n-1)^2$ coefficients that define the strength of a suitable independent set of elementary basis matrices. At the same time this feature gives us a chance to choose a set of basis matrices that simplifies the interpretation of phenomena.

The presence of $\mathbf{G}^{(\mathrm{z})}$ prevents the existence of a potential matrix. Conversely, in the absence of this type of interaction a potential matrix can be derived and thermodynamic behavior can be observed in multiagent living systems following suitable dynamical rules. Within the decomposition framework presented above, the potential matrix can be easily determined, exploiting the symmetry properties of the game components. The potential matrix of symmetric games is symmetric, and it always exists when $n = 2$. In contrast, the potential matrix of a bimatrix game can also include antisymmetric components when the game has finite directed anticoordination or zero-sum self- and cross-dependent payoff components. The numerical investigation of a three-strategy spatial evolutionary game with this curious interaction on a square lattice shed light on the preservation of power law behavior in the vicinity of the critical phase transition while deviations occur in the macroscopic behavior. The clarification of other curious consequences when this interaction is combined with other orthogonal components requires further systematic investigations.

The systematic consideration of orthogonality conditions has proved to be fruitful and helped the identification of relationships occurring in many other fields of science where the mathematical background is based on the application of matrices. The most striking feature is that the orthogonality between the original game and one of the types of interactions indicates said type's absence from the interaction and the appearance of symmetries and related consequences in the macroscopic behavior for some dynamical rules that can be easily identified. Additionally, the feature of orthogonality can be exploited for the quantification of the proportions of the different types of interaction, as shown by Eq. (10) for symmetric games. The generalization of these measures to bimatrix games is trivial and unavoidable for the prospective consideration of coevolutionary games in which the evolution of interactions through consecutive mutations is allowed.

CRediT authorship contribution statement

György Szabó: Writing – original draft, Visualization, Software, Investigation, Formal analysis, Conceptualization. **Balázs Király:** Writing – review & editing, Investigation, Formal analysis.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Data availability

Data will be made available on request.

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