Competing associations Lecture 13

literature: Szabó and Fáth, *Phys. Rep.* 446 (2007) 97

What happens in spatial games with many strategies?

Locally only a few strategies can be present at the same time.

=> Evolutionary rule finds optimum in the corresponding subgames.

=> Different solutions are formed and distributed in the system.

=> These solutions (as strategy associations) compete for territory.

=> These associations can be considered as complex objects.

These phenomena can be studied within the framework of

spatial predator–prey models.

These models are simple and can be studied with high efficiency.

The number of possible solutions and associations increases exponentially with the number of species.

Predator–prey models on the square lattice

Individuals of n species are located at the sites of the lattice.

Each site x is singly occupied and characterized by the species occupying it.

$s_x = 0, 1, \ldots, n-1$

The predator–prey relation is defined by a food web.

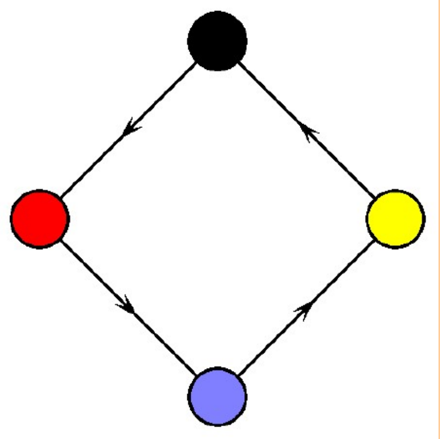
The evolution is started from a random initial distribution of species.

Elementary steps (invasion and site exchange) are repeated:

- We choose two neighbouring sites (x and y) at random.
- The (s_x,s_y) pair transforms into (s_x,s_x) if s_x is the predatot of s_y ;
  (s_y,s_y) if s_x is the prey of s_y ;
  (s_y,s_x) with a prob. X_s if s_x and s_y are neutral.

The stationary state can be described by the composition (frequency of species) and/or correlation functions.

Four-species cyclic predator–prey model



s_x=0, 1, 2, 3

Population dynamics:

Frequency of species j: ρ_j

Equations of motion:

$$\dot{\rho}_1 = \rho_1(\rho_2 - \rho_4)$$
$$\dot{\rho}_2 = \rho_2(\rho_3 - \rho_1)$$
$$\dot{\rho}_3 = \rho_3(\rho_4 - \rho_2)$$
$$\dot{\rho}_4 = \rho_4(\rho_1 - \rho_3)$$

Conserved quantities: $\sum_j \rho_j = 1, \qquad \prod_j \rho_j = \text{const}$

Trivial stationary solutions:

0<c<1: arbitrary constant

$$\rho_1 = 1, \quad \rho_2 = 0, \quad \rho_3 = 0, \quad \rho_4 = 0;$$
$$\rho_1 = 0, \quad \rho_2 = 1, \quad \rho_3 = 0, \quad \rho_4 = 0;$$
$$\rho_1 = 0, \quad \rho_2 = 0, \quad \rho_3 = 1, \quad \rho_4 = 0;$$
$$\rho_1 = 0, \quad \rho_2 = 0, \quad \rho_3 = 0, \quad \rho_4 = 1;$$
$$\rho_1 = c, \quad \rho_2 = 0, \quad \rho_3 = 1 - c, \quad \rho_4 = 0;$$
$$\rho_1 = 0, \quad \rho_2 = c, \quad \rho_3 = 0, \quad \rho_4 = 1 - c;$$
$$\rho_1 = \rho_2 = \rho_3 = \rho_4 = 1/4;$$

Four-species cyclic predator-prey model

Simulations: two phases (from random initial state)

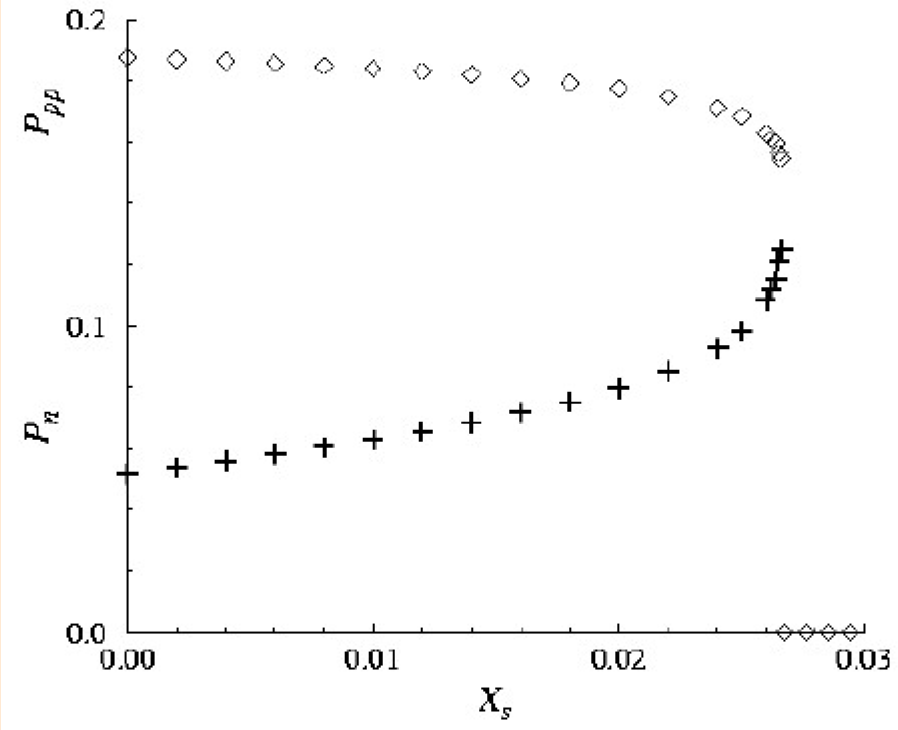
If $X_s < X_c$, then cyclic invasions, (0+1+2+3).

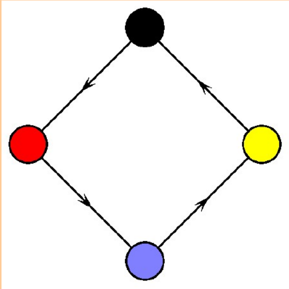
If $X_s > X_c$, then mixture of the two neutral species, (0+2) or (1+3).

X_c=0.02665(4)

p_{pp} prob. of predator–prey and

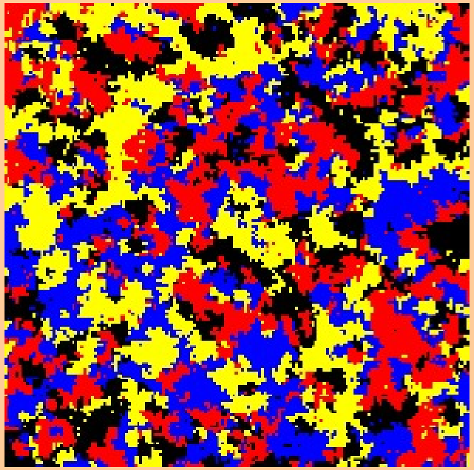
p_n prob. of neutral pairs as a function of X_s:





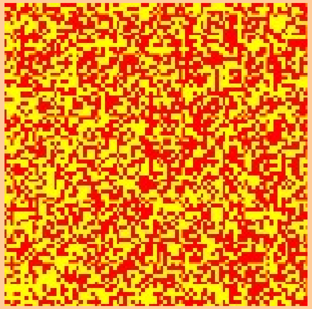
Self-organized state

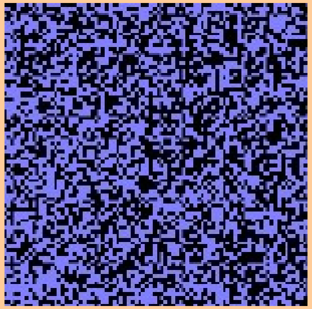
X=0



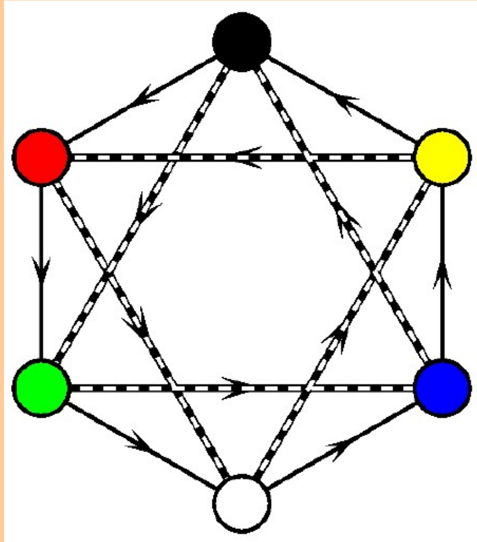
Defensive association of two neutral species:

X=0.15





Six-species predator–prey model



Each species has two prey and two predators.

Predator–prey relations are given by the food web:

This directed graph contains 3-, 4-, and 5-species loops.

There are three neutral pairs (e.g., black and white).

Possible (subsystem) solutions:

- one-species (homogeneous, e.g., s_x=2, invadable by two others)
- mixture of two neutral pairs (e.g., black and white with arbitrary ratio)

  Their well-mixed state is a defensive alliance.
- cyclic triplet (e.g, red+yellow+white, playing a rock–paper–scissors game)

  These are also defensive alliances, stabilized by the proper self-organizing pattern.
- cyclic four-species state (e.g., red+green+blue+yellow)
- five- and six-species solutions

Question: Which one will be the winner as a function of X_s? …

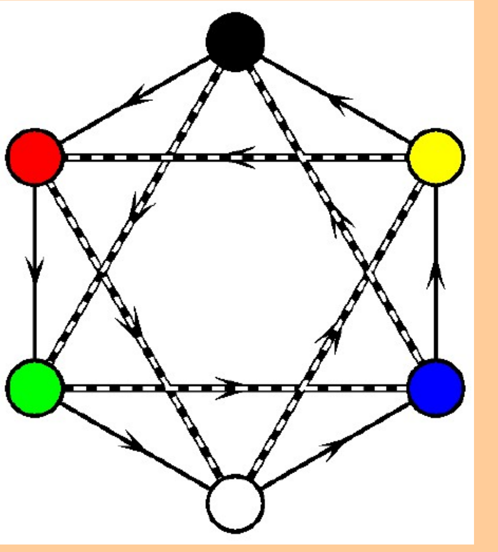
Simulations: growing domains

If $X_s < X_c$, then (0+2+4) or (1+3+5).

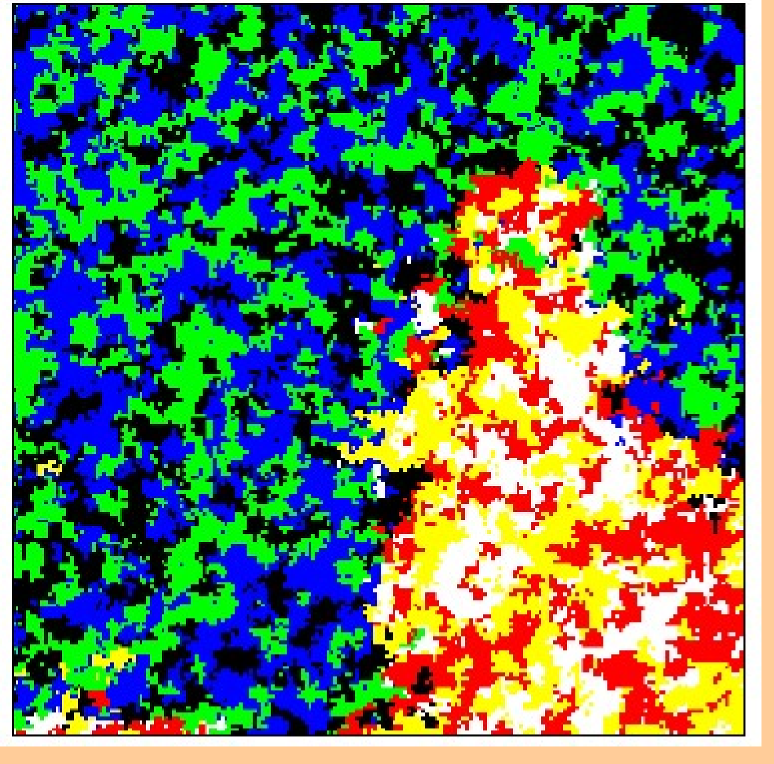
If $X_s > X_c$, then (0+3), (1+4), or (2+5).

$$X_c = 0.05592(1)$$

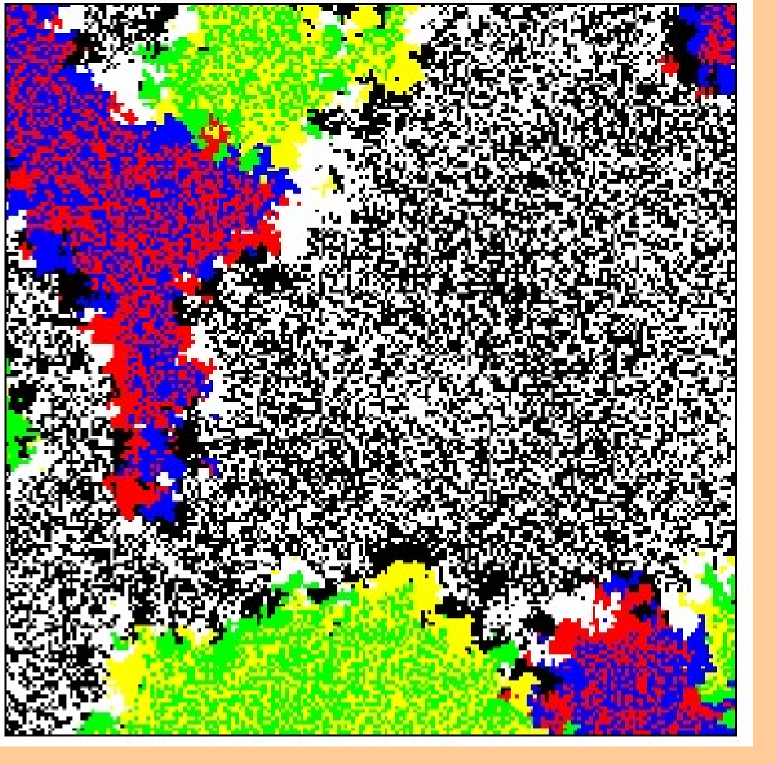
Spatial distribution of species:



X=0



X=0.15



Warfare of bacteria with two toxins

Bacteria use toxins to kill each other:

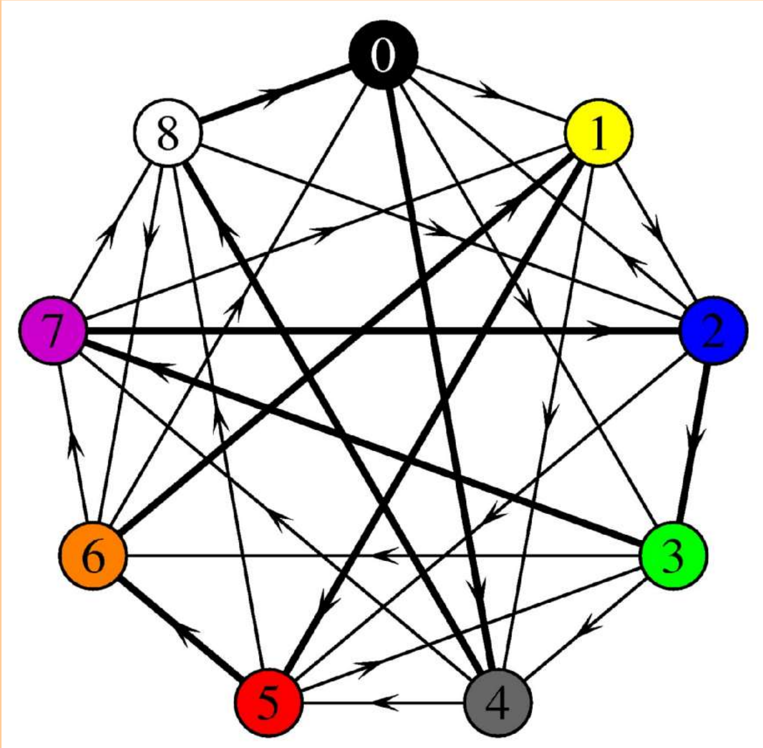
T: toxic (produces toxin- and anti-toxin)

R: resistant (produces only anti-toxin)

S: sensitive

$T \rightarrow S \rightarrow R \rightarrow T$

rock–paper–scissors-type cyclic dominance



For **two toxins**, 9 species can be distinguished:

TT, *TS*, *TR*, *ST*, *SS*, *SR*, *RT*, *RS*, *RR*

Each phenotype has three predators and three prey.

There are many cycles within the food web.

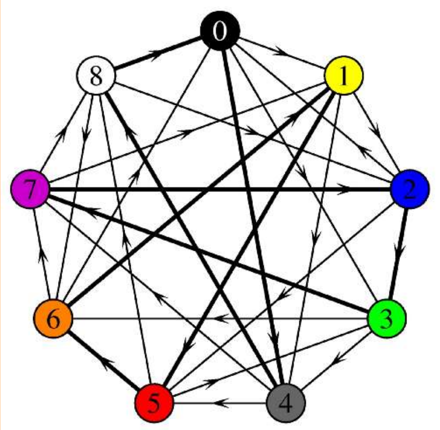
Thick triangles: cyclic defensive alliances

Simulations

Slow mixing (small X)

Growing domains:

Three cyclic defensive alliances

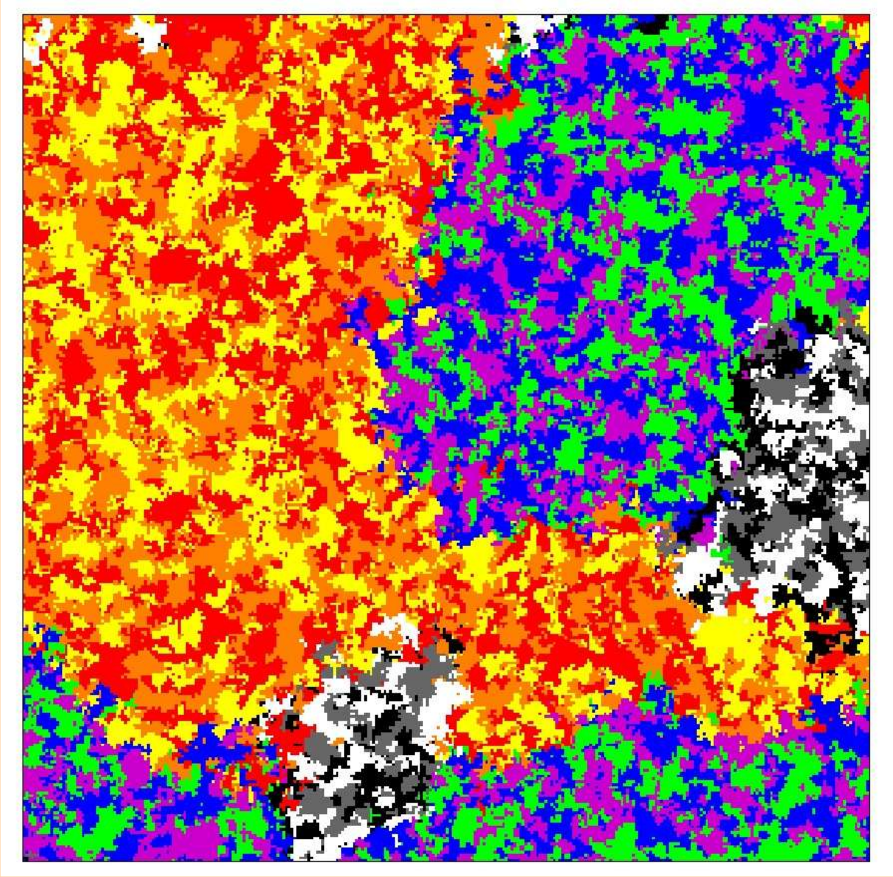


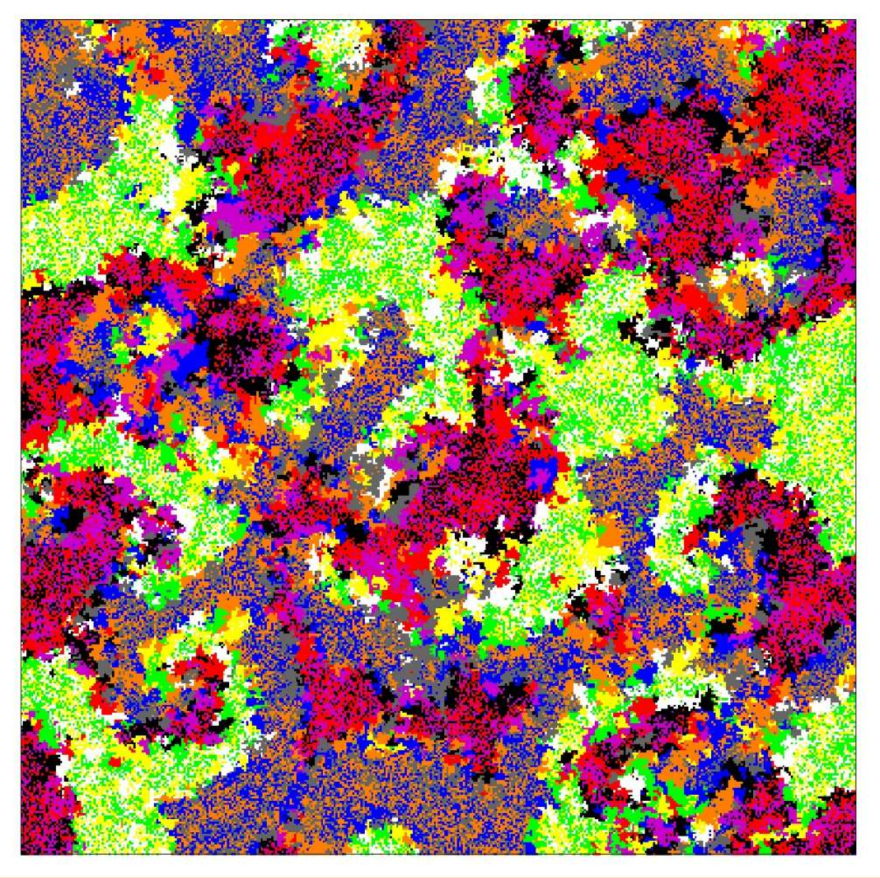
Fast mixing

Well-mixed neutral triplets
play a rock–paper–scissors game.

X=0

X=0.2





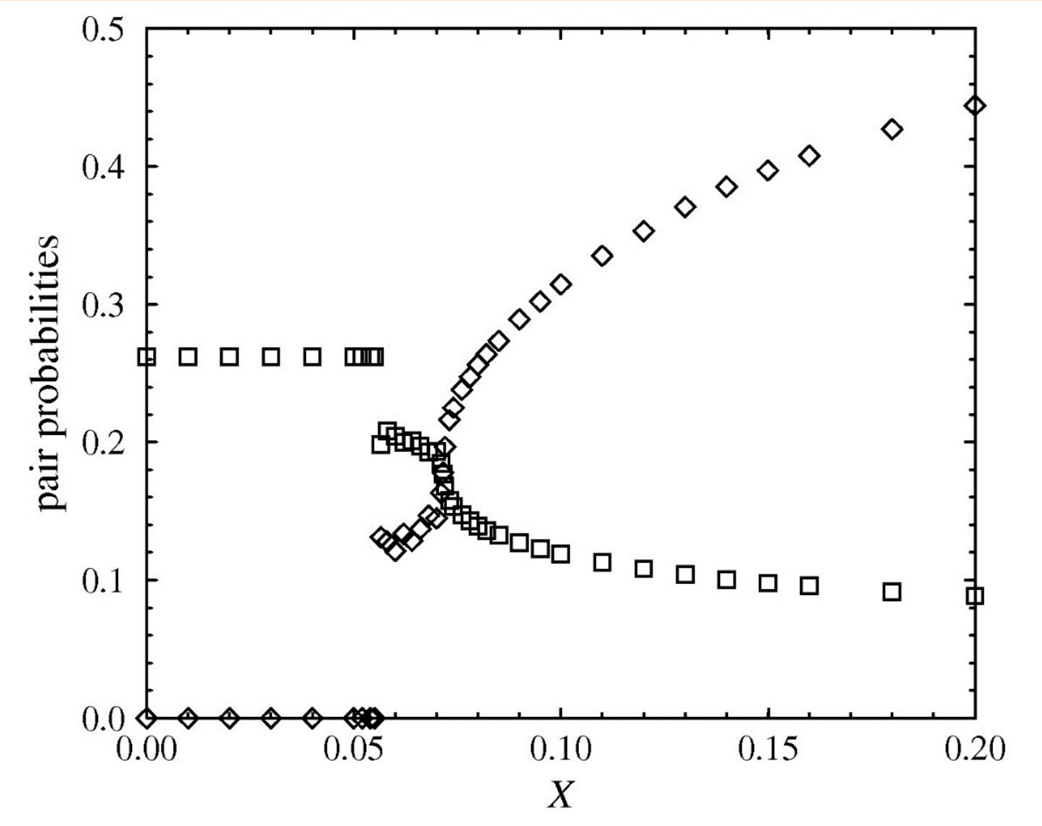
Numerical results

Three options on two neigbouring sites: - same species

- neutral pair

- predator–prey pair

Probability of finding neutral (◊) and predator-prey (□) pairs vs. X



Three states:

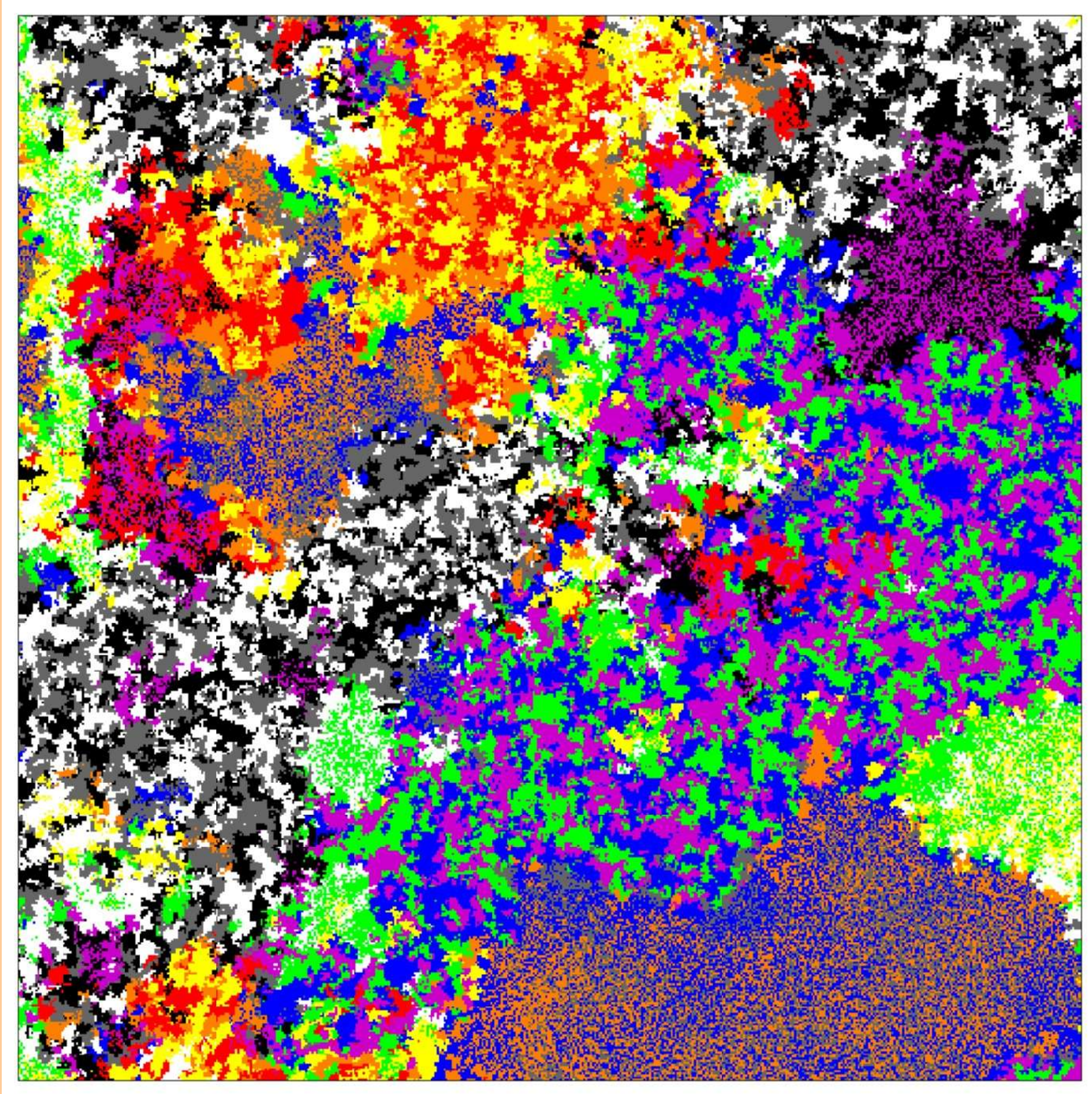
$X<X_{c1}$: cyclic defensive alliances

$X>X_{c2}$: well-mixed neutral triplets play RSP game

$X_{c1}<X<X_{c2}$: more complex patterns, huge fluctuations, avalanches

$X_{c1}=0.056$, $X_{c2}=0.072$:

Snapshot (size: 500x500)



X=0.066

L=2000 (total size)

t=10,000 MCS

Coexistence of cyclic and neutral triplets

+

Two-species neutral mixed states can grow up slowly and become extinct fast.

”Phoenix bird effect”

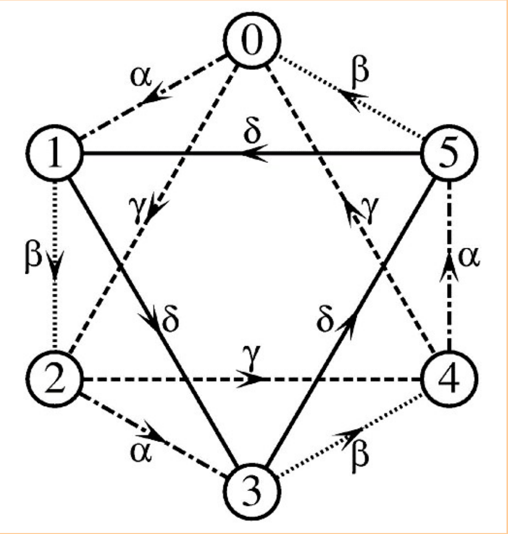
Varying invasion rates

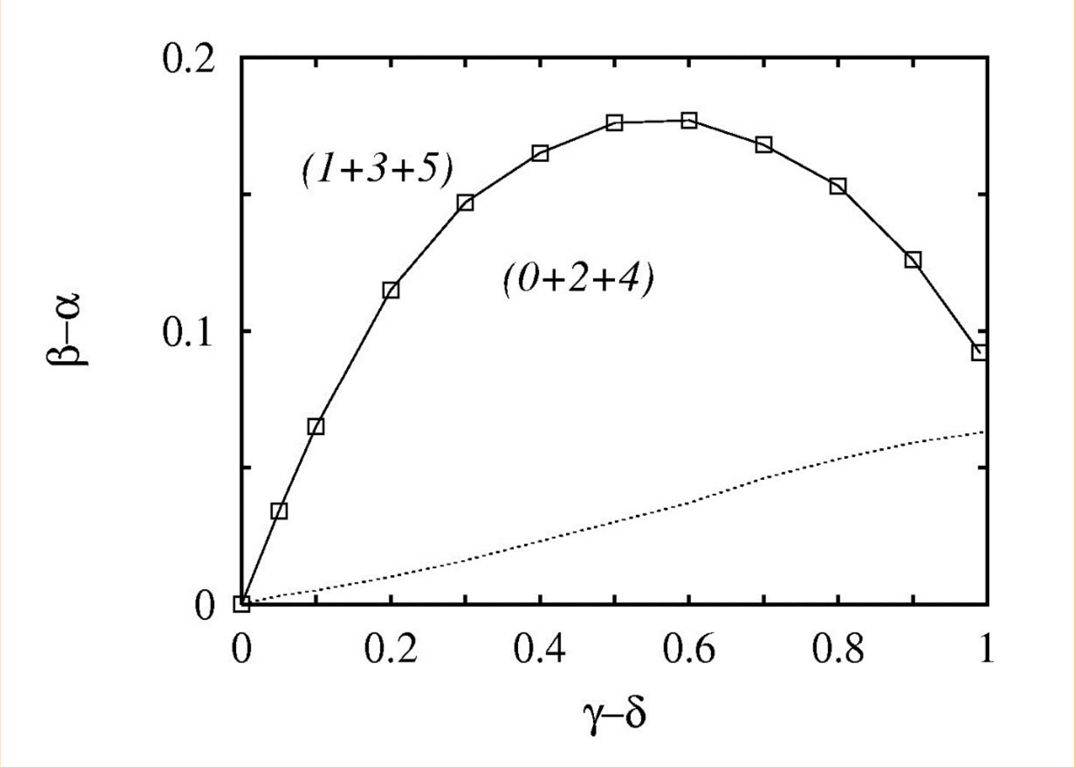
The six-species model is extended by introducing four different invasion rates.

Three-fold symmetry is conserved.

Simulation results: □

4-site approximation: ---------- if X=0



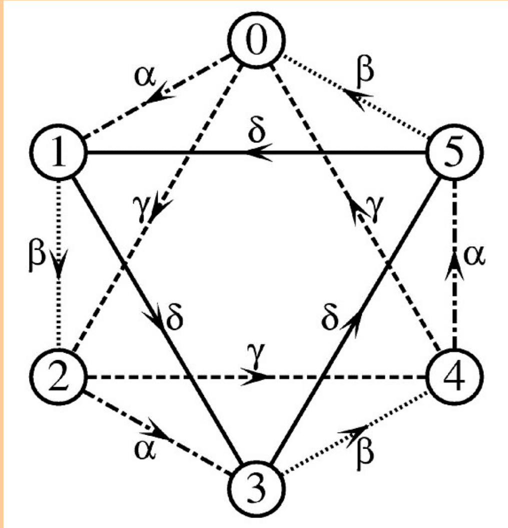


Faster internal cycle stabilizes its triplet more efficiently.

Behaviour under mixing (X>0)

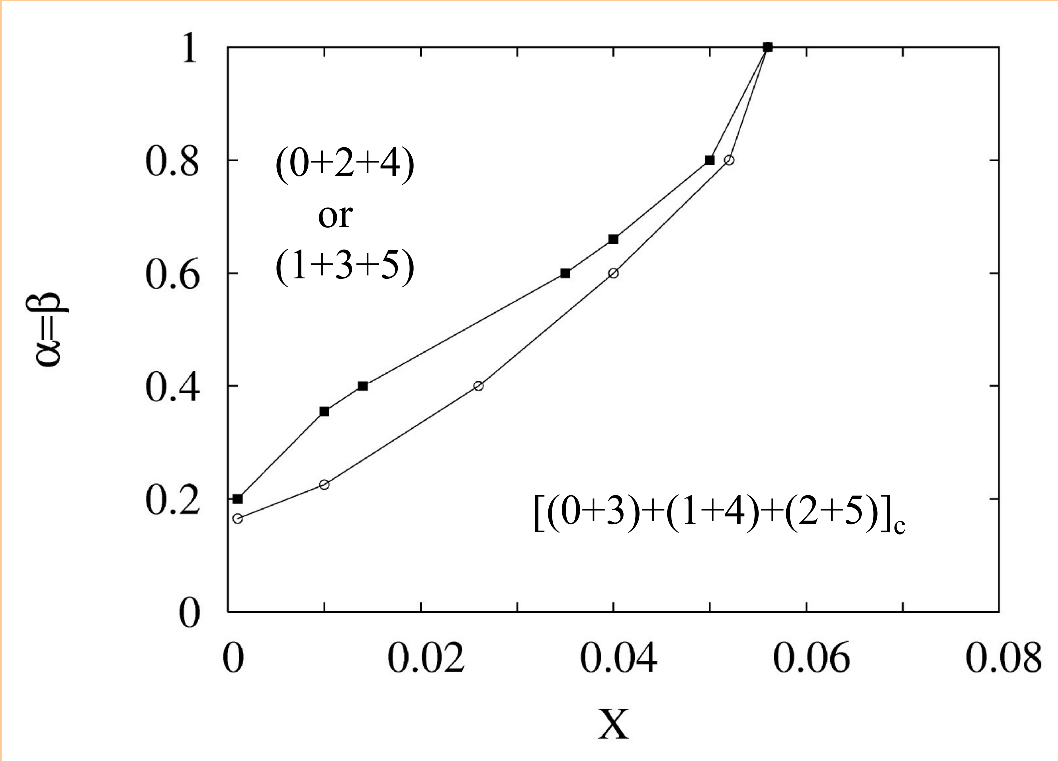
Cyclic dominance between neutral pairs
if $\alpha>\delta$ and $\beta>\gamma$, or in the opposite case.

Simulation: $\alpha=\beta=0.75$ and $\gamma=\delta=1$, $X=0.15$



simulation

The phase diagram indicates complex behaviour inside the intermediate region.

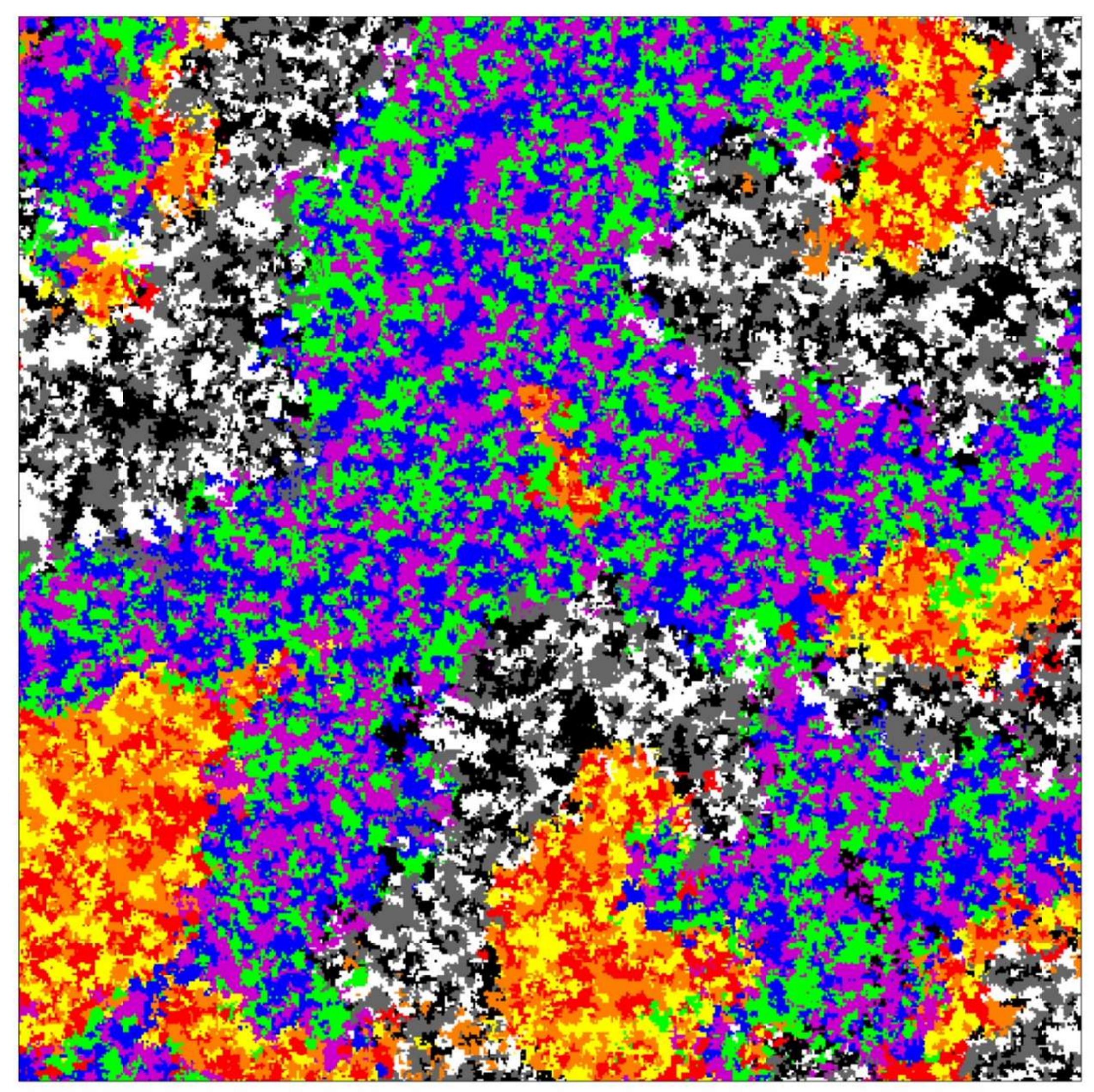


Invasion inhomogeneity in the two-toxin model

Invasion rates are decreased in one of the toxin cycles. simulation

Consequence:

Cyclic dominance among the cyclic triplets.



Chimera states

The chimera is a monster from Greek mythology.

Chimera states are polydomain structures in spatial clock models, in which the clocks are synchronized within the domains.

Discretized versions of these behaviours can be well reproduced by rock–paper–scissors models on a square lattice when the logit rule is applied synchronously.

It is a stochastic cellular automaton (t=0,1,2,…).

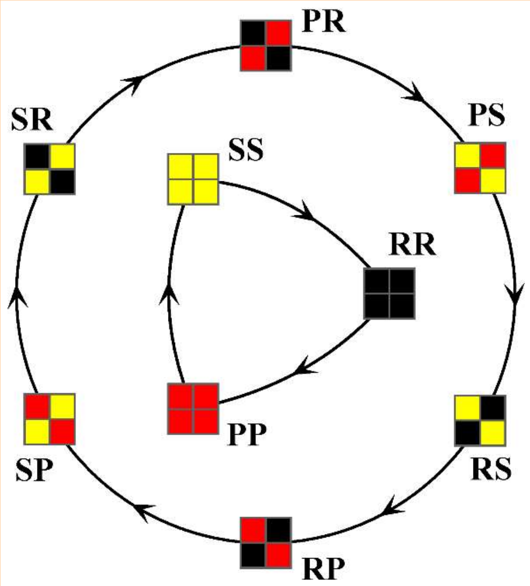
Simulations:

t=0, 1, 2, 3, …   movie7

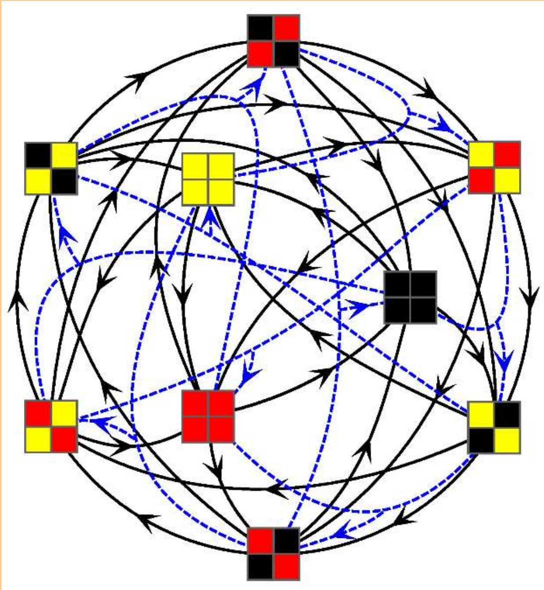
T=0, 6, 12, …   movie8

The self-organizing patterns are maintained by cyclic competitions between the oscillating phases and by the appearance of new phases along some interfaces (e.g.: PR+SS →PS).

Cycles



Dominance



Summary

Increasing the number of strategies results in the spontaneous formation of strategy associations with appropriate spatio-temporal patterns.

The associations can be considered as complex objects playing games against each other.

Defensive alliances are preferred associations, as their components guard each other against external invaders. It is a type of cooperation.

Cyclic dominance appears frequently and supports the survival and coexistence of a large number of individual strategies ((bio)diversity).

Similar behaviour is characteristic of biological and social systems.

Evolutionary game theory is capable of describing such complex behaviours, too.

Home assignments

13.1. The six-species model introduced on page 5 is defined by a food web that is a directed graph. A directed graph can be characterized by its adjacency matrix **A** whose elements A_{ij} are 0 if species i and j are neutral (not linked); A_{ij}=1 if there is a link directed from i to j; and A_{ij}=–1 if there is a link from j to i. Determine the adjacency matrix for this system! (Note: This adjacency matrix can be considered as the payoff matrix for this six-strategy game.)

13.2. What are the conserved quantities in the population dynamics of the model mentioned above?