Spatial evolutionary games with imitation Lecture 9

literature: Szabó and Fáth, *Phys. Rep.* 446 (2007) 97

N players are located at the sites of a lattice (with periodic boundary conditions).
The player at site *x* follows one of the *n* pure strategies represented by unit vectors:

$$\mathbf{s}_x = \begin{pmatrix} 1 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad \begin{pmatrix} 0 \\ \vdots \\ 1 \end{pmatrix}.$$

Player *x* plays games with her neighbours, her total payoff is

$$u_x = \sum_{\delta} \mathbf{s}_x^{+} \mathbf{A} \mathbf{s}_{x+\delta} = \sum_{\delta} \begin{pmatrix} s_{x1} & \cdots & s_{xn} \end{pmatrix} \begin{pmatrix} A_{11} & \cdots & A_{1n} \\ \vdots & \ddots & \vdots \\ A_{n1} & \cdots & A_{nn} \end{pmatrix} \begin{pmatrix} s_{x+\delta,1} \\ \vdots \\ s_{x+\delta,n} \end{pmatrix}$$

Assumptions:

- The players are equivalent and use the same pure strategy in their games against neighbouring players at sites $x+\delta$.
- The games are uniform.
- The games are repeated and the players can modify their strategy to increase their own income.
- We are (mostly) interested in the stationary state.

Evolutionary rules for repeated spatial games

Each player can modify her strategy.
The selection of a new strategy depends on the payoff(s).
Many kinds of rules exist:

synchronized or random sequential strategy update
deterministic or stochastic
additional capabilities (e.g., memory) can be introduced for the players
local or global, etc.

For potential games and Glauber-type evolutionary rules, the system evolves into the Boltzmann distribution, thus allowing the application of thermodynamics, as detailed in Lecture 8.
In that case player x modifies her strategy $\mathbf{s}_x$ to $\mathbf{s}'_x$ via the logit rule with probability:

$$w(\mathbf{s}_x \to \mathbf{s}'_x) = \frac{\exp[u_x(s'_x)/K]}{\sum_{s''_x} \exp[u_x(s''_x)/K]}$$

Interpretation of K: - temperature

- amplitude of noise

- magnitude of risk, etc.

Evolutionary social dilemmas on a square lattice:

The first spatial model was studied by Nowak and May, 1993.

Two strategies: *C* (cooperator in the PD game)

D (defector in the PD game)

$$D = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad C = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Players collect income from games played with their *z* neighbours.

Scaled payoff matrix:

$$A = \begin{pmatrix} 0 & T \\ S & 1 \end{pmatrix}, \quad 1 < T = b < 2 - S, \quad S \le 0.$$

Deterministic and synchronized evolutionary rule

cellular automaton on a square lattice for S=0

first and second neighbour interactions (including self-interaction) (z=9)

discrete time steps: t=0, 1, 2, …

At time t the strategy distribution is $\mathbf{s}_x(t)$,

and the payoff distribution is $u_x(t)$, which depends on $s_x(t)$.

At time t+1 each player adopts the strategy of her most successful neighbour

$$s_x(t+1) = f[s_x(t), \{s_{x+\delta}(t)\}].$$

Simulations

Solitary *D* in a sea of *C*s

Random initial state

b=1.85

b=1.1 *b*=1.31 *b*=1.56 *b*=1.81

Results of simulations

frequency of C: ρ

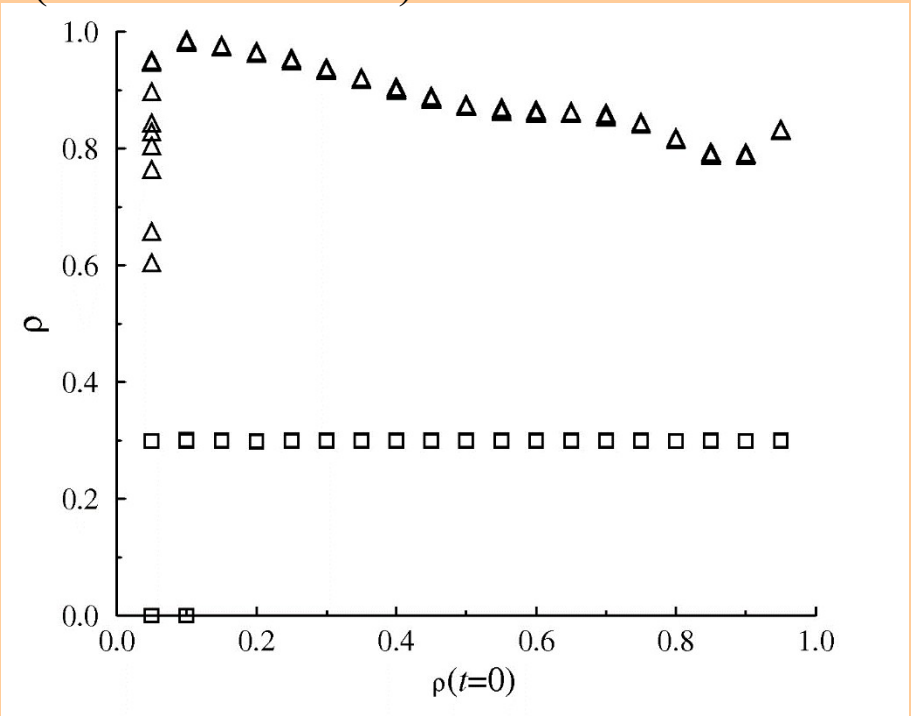
with or without self-interaction at z=8(□) or 9(◊) [self-interaction supports C]

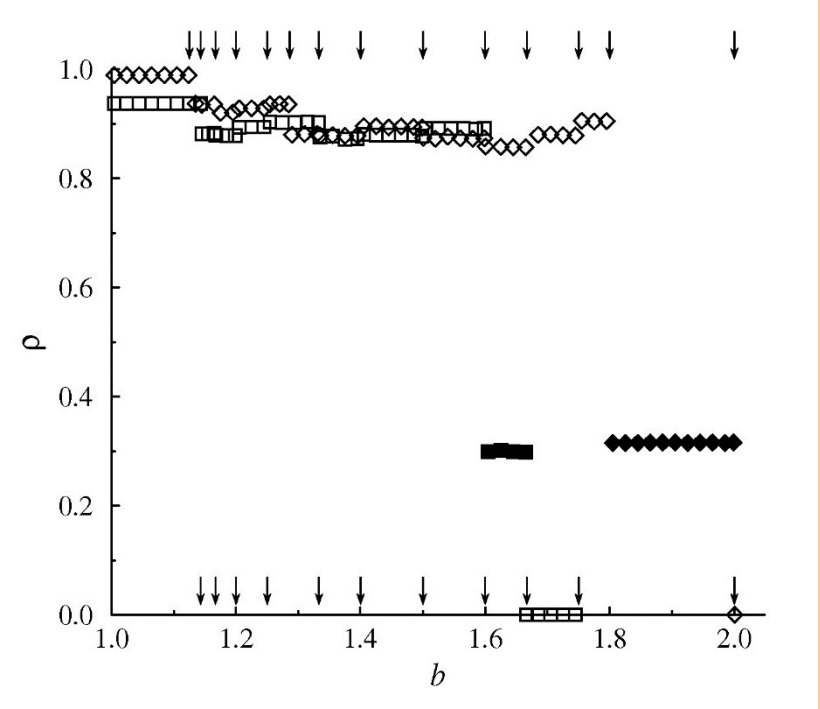
b-dependence [$\rho(t=0)$=1/2]

open symbols: limit cycles

closed symbols: ” chaotic” behaviour

dependence on the initial ρ at b=1.35 and 1.65 (no self-interaction)





sensitivity to initial configuration

break points in b-dependence

Messages deduced from the model of Nowak and May:

- Maintenance of *C* is supported: by invasion of *C* along straight boundaries,
  by the formation of rectangular *C* colonies.
- Irregular interfaces support *D* invasions.
- Coexistence of *C* and *D* can occur.
- Appearance of frozen patterns or limit cycles.

Shortcomings and difficulties:

- artificial patterns characteristic of cellular automata

  S. Wolfram distinguished four types of patterns:

  - homogeneous states
  - frozen or oscillating patterns
  - random behaviour
  - distributions with long-range correlations

- weak noise influences the final state significantly
- analytical treatment is difficult

Imitation based on random pair comparison

- We choose a nearest-neighbour pair (x,y) at random.
- Payoffs are determined for both players: u_x, u_y.
- Player x adopts their neighbour's strategy with probability

$$w(s_x \to s_y) = \frac{1}{1+\exp[(u_x - u_y)/K]},$$

where K denotes the magnitude of uncertainty (noise).
sources of noise: payoff fluctuation, errors in decision-making, free will, etc.

- This rule can drive the system into an absorbing state:
  Homogeneous states remain unchanged.
- If the system started from a random state, then it can evolve into a stationary state that is independent of the initial state.
- Stationary states are characterized by: the frequency of strategies,
  correlation functions,
  correlation lengths, etc.

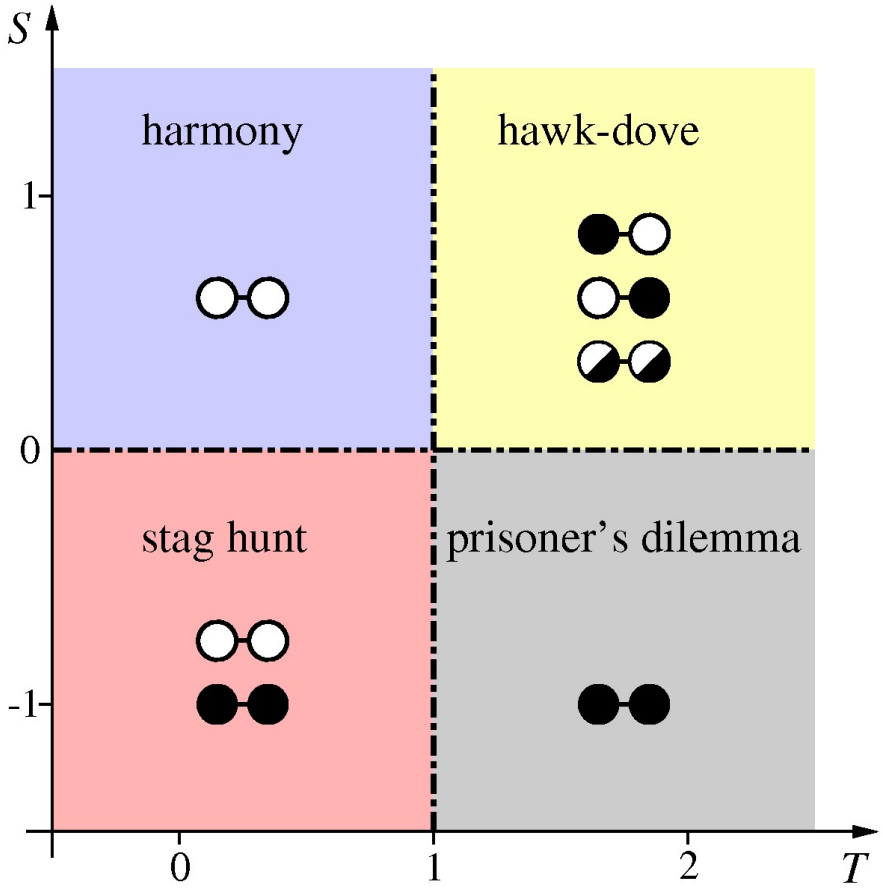
Result of replicator dynamics for social dilemmas: $\mathbf{A} = \begin{pmatrix} 0 & T \\ S & 1 \end{pmatrix}$

The mean-field approximated model can be solved for arbitrary S and T values:

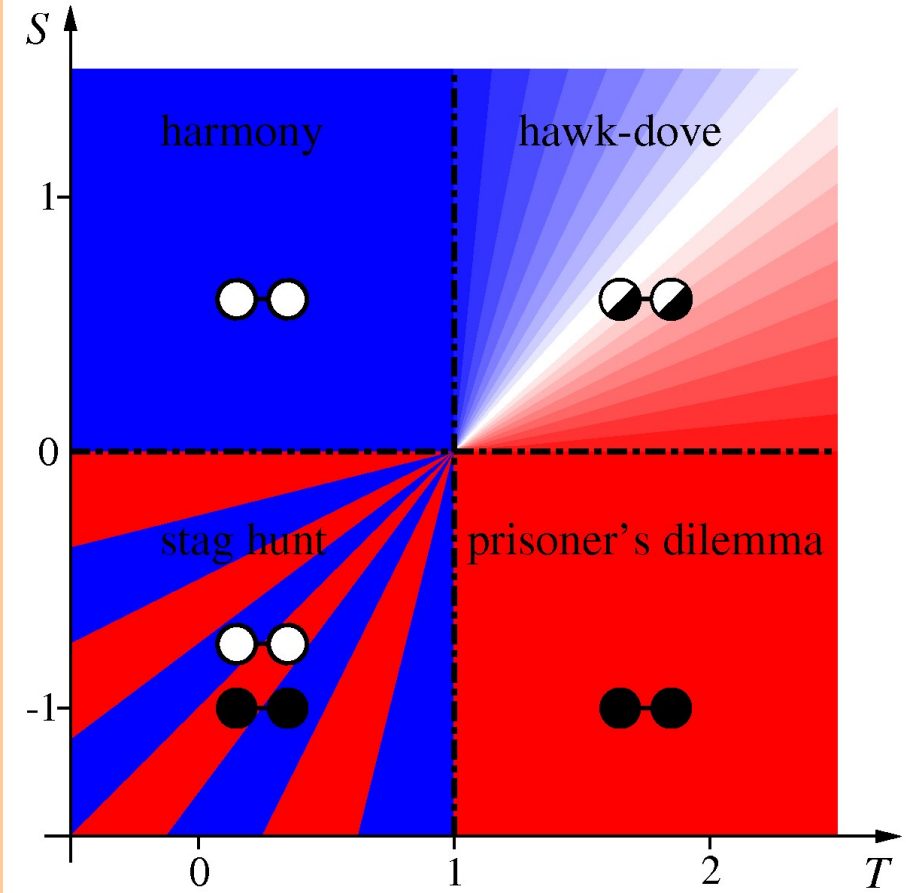
Four types: 1) harmony: ρ=1; 2) hawk-dove: $0 < \rho < 1$

3) stag hunt: ρ=0 or 1 4) prisoner's dilemma: ρ=0

Nash equilibria:



stationary solutions:



Results of Monte Carlo simulations on the square lattice for z=4

Three types of coexistence (R=1, T=b, S=0, P=0):

1. 'solitary' Ds, analogous to uniting-branching random walk — b=0.95
2. $\rho \approx 0.5$ — b=1.01
3. C colonies perform uniting-annihilating-branching random walks — b=1.035

Results of extended analytical approaches: red line: MF approximation;

dotted: pair approximation; dashed: 4-site approximation; solid: 9-site approximation

Numerical results, if K=0.4

Two **critical transitions** at b_{c1} and b_{c2}

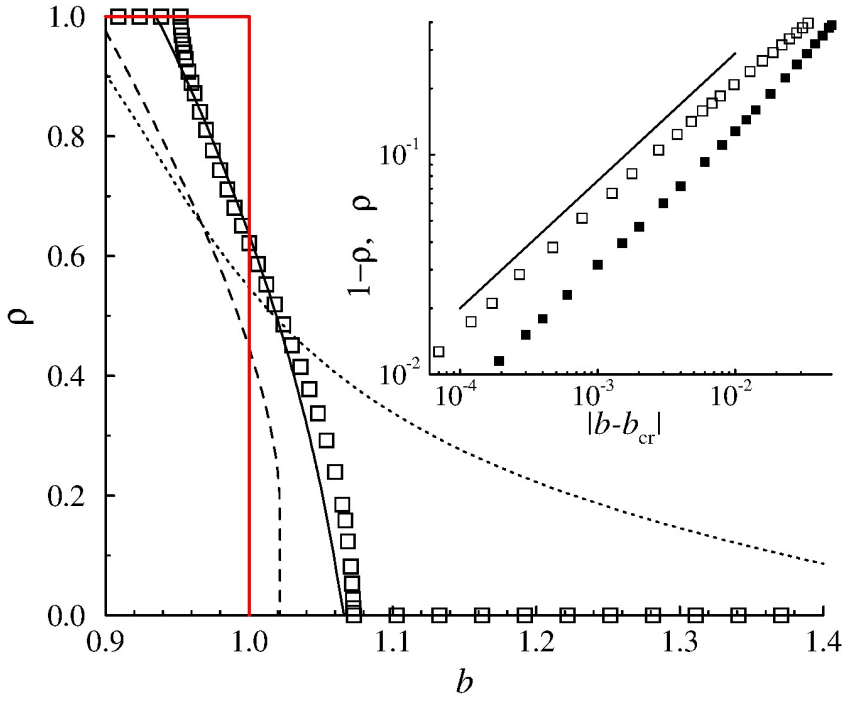
Extinctions of C or D are universal:

„Directed percolation" universality class

$$\rho \approx \left| b - b_{cr} \right|^{\beta}, \quad \beta = 0.58(1)$$

Divergence in fluctuations, correlation length and time.

Difficulties in simulations.

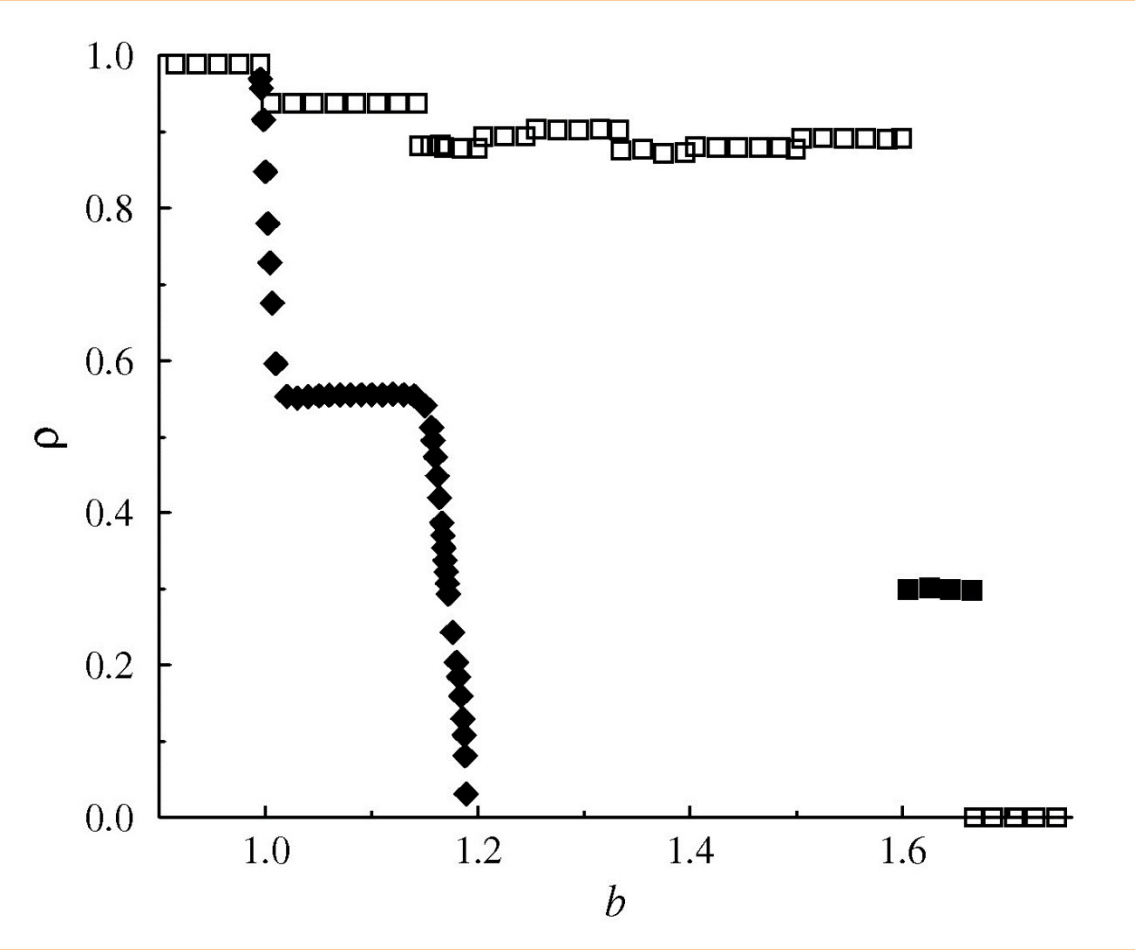


Comparison of the results obtained for different evolutionary rules

z=8, first- and second-neighbour interactions

- NM cellular automaton (□)

- comparison of random pairs at K=0.03 (♦)



NM cellular automaton

step like b-dependence

ρ and b_{c2} are larger

Random pair comparison

continuous variation in ρ

Stochasticity reduces the survival chances of C.

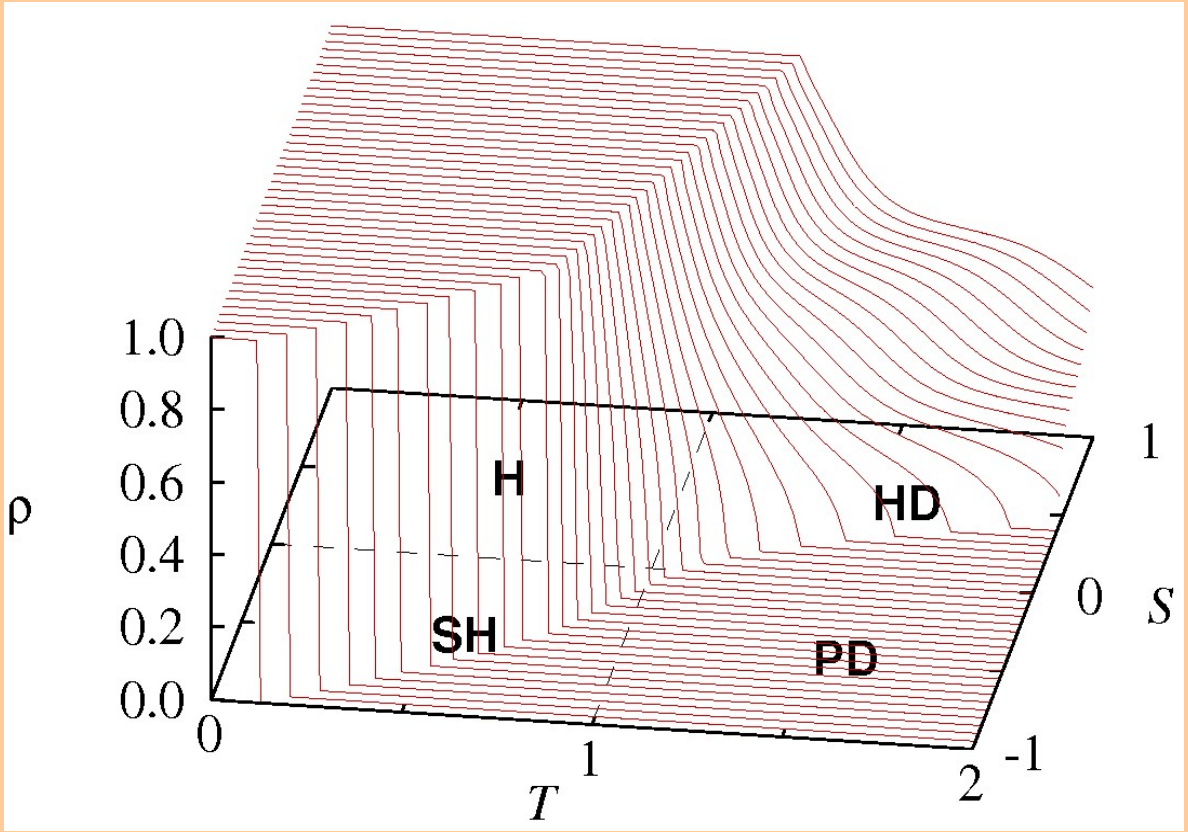
Social dilemmas on the square lattice for random pairwise imitation (z=4)

Payoff matrix:

$$\mathbf{A} = \begin{pmatrix} 0 & T \\ S & 1 \end{pmatrix}$$

Simulations for K=0.25:

Frequency of C: ρ



Absorbing states at: ρ=0 and 1

DP transition if, $\rho \rightarrow 0$ or 1

First-order transition in the SH region.

For lower noise levels in the SH region we can observe both the ρ=0 and 1 states.

The final results depend on the initial state and randomness.

Anisotropic invasion and its consequences for imitation

The payoff matrix:

$$\mathbf{A}=\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}+\varepsilon\begin{pmatrix}1 & 1\\ -1 & -1\end{pmatrix}+\delta\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix}=\begin{pmatrix}1+\varepsilon+\delta & -1+\varepsilon-\delta\\ -1-\varepsilon+\delta & 1-\varepsilon-\delta\end{pmatrix}$$

coordination self-dependent cross-dependent (for games)

Ising coupling ext. magn. field --- (interaction in physical systems)

Dynamics: stochastic imitation of the neighbour's strategy

$$w(s_x \to s_y)=\frac{1}{1+\exp[(u_x-u_y)/K]}$$

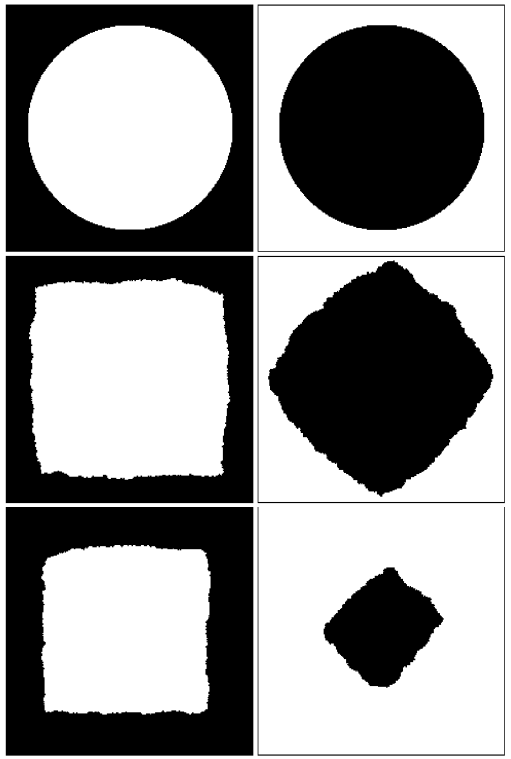
Evolution of circular domains, if δ=0.5, ε=–0.03, K=0.3, and r=90:

Black and white colors represent strategy 1 and strategy 2

white **phenomenon**:

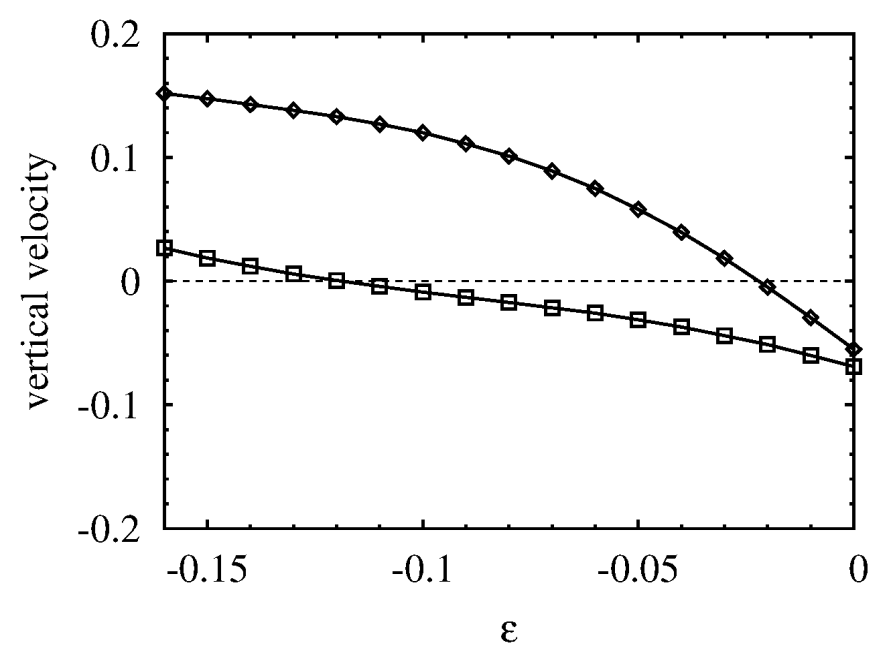
The velocity of the invasion front depends on its orientation.

black Opposite net directions for horizontal (or vertical) and tilted interfaces (with ±45º).



Velocity of invasion versus ε

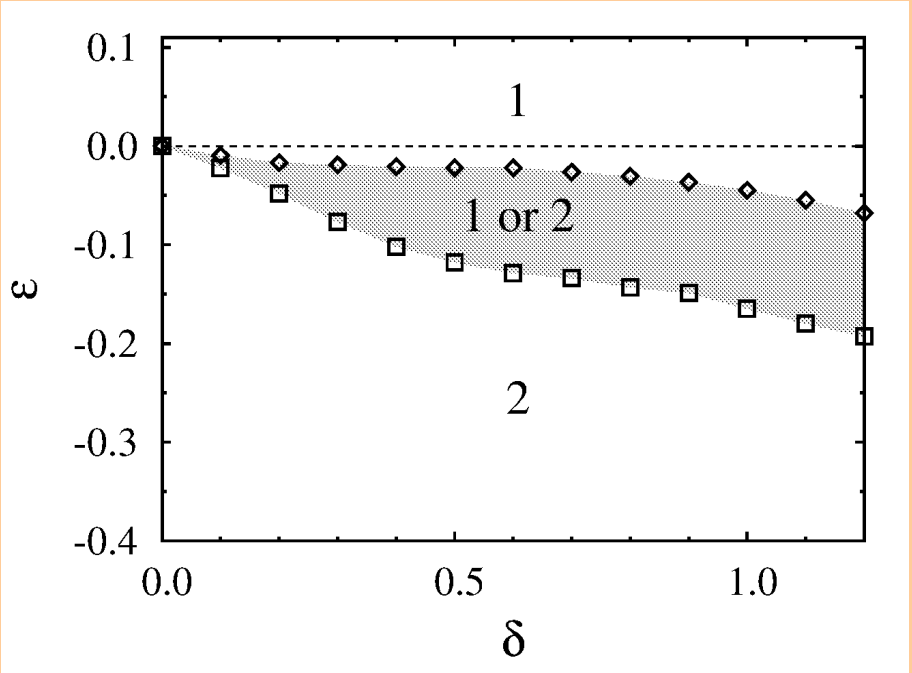
MC simulations with strip-like initial states (δ=0.5, K=0.3, L=1000, averaging over 100 runs)



Symbol edges correspond to interface orientations.

There is a parameter region (grey) in which opposite signed interface velocities occur.

Here the system can evolve into both types of homogeneous states.



This phenomenon is robust.

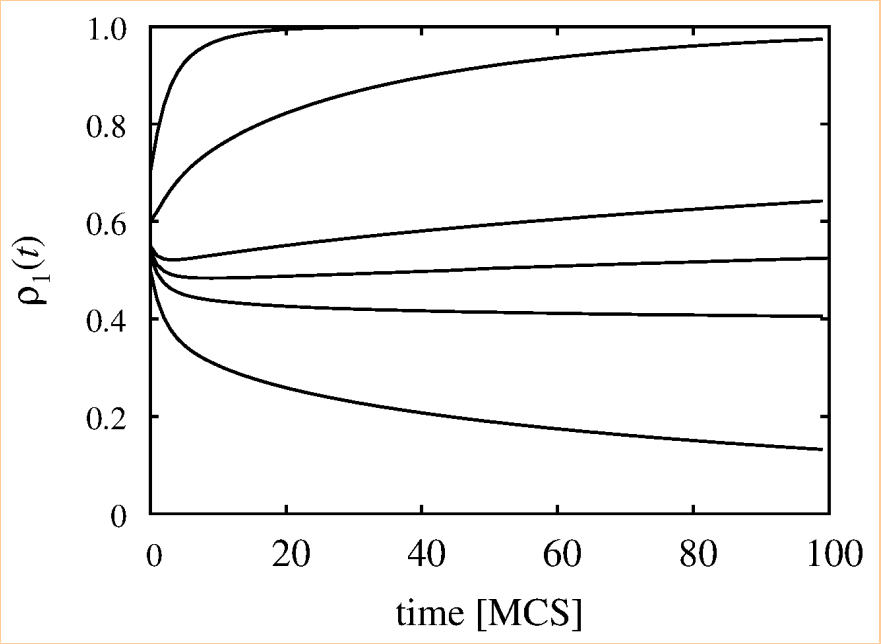
It can occur on other lattices at low noises.

The selection of the final state can be explained by the theory of continuous percolation.

Percolation (periodic boundary conditions are assumed)

Two-color maps (patterns) include a (percolating) sea in which there are islands with many lakes in which there are smaller islands, etc. For opposite invasion velocities, both types of islands (or lakes) shrink and vanish. Thus, the type of the percolating sea will determine the final homogeneous state. The percolating phase is determined by the composition of the initial state and also by the early evolutionary processes, which are better described by mean-field approaches in the absence of correlations in the initial state.

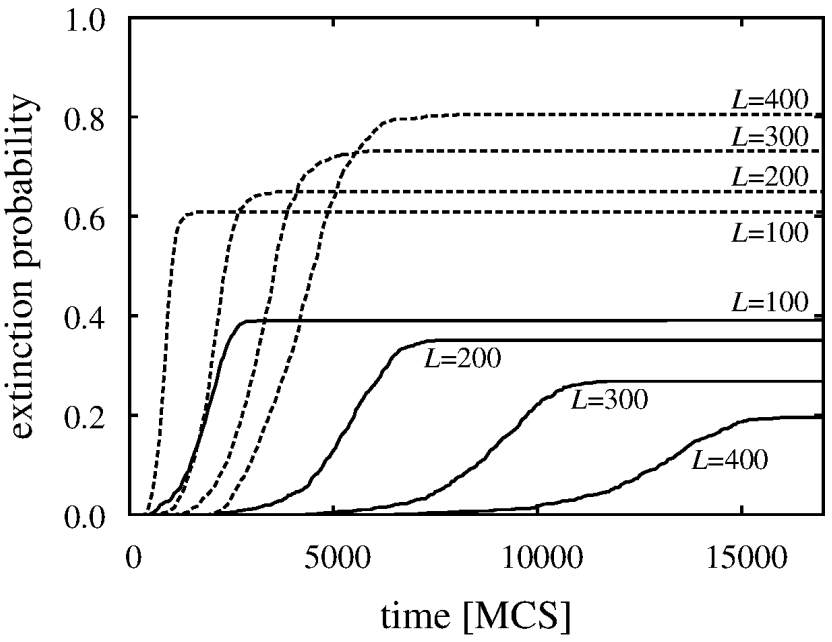
Strategy frequency vs. time for δ=0.5, ε=–0.03, K=0.3, L=2000



if $\rho_1(0)$=0.7, 0.6, 0.55, 0.54, 0.53, and 0.5

$\rho_1(t) \to 0$ or 1, if $t \to \infty$

(averaging over 1000 runs)



Extinction of strategies for different linear sizes when $\rho_1(0)$=0.54.

Home assignments

9.1. Evaluate the typical payoffs in the Nowak–May cellular automaton for $z=8$ when the domains of cooperators and defectors are separated by a horizontal boundary!

9.2. What are the payoffs in the previous problem if one of the players along the boundary reverses her strategy?

9.3. What are the strategy distributions at $t+1$ if the spatial distribution at t is the same as in 9.1. and 9.2. for $b=1.3$ and 1.7?