Thermodynamical behaviour in evolutionary games

If in a multiagent evolutionary game the interactions are defined by two-person potential games and the evolution is controlled by the logit rule, then the system evolves into the Boltzmann distribution that ensures the validity of statistical physics and the laws of thermodynamics.

These systems have equivalent players located at the sites (x=1, …, N) of a lattice or graph, and the players use one of their available strategies (s_x=1, …, n) against all of their neighbours (at sites $x+\delta$) that results in a total payoff of

$$u_x(s_x) = \sum_{\delta} \mathbf{s}_x \cdot \mathbf{A}\mathbf{s}_{x+\delta}$$

In these systems, the potential as a function of $\mathbf{s}=(s_1, \ldots, s_N)$ can be expressed as:

$$U(\mathbf{s}) = \frac{1}{2}\sum_{x,\delta} \mathbf{s}_x \cdot \mathbf{V}\mathbf{s}_{x+\delta}$$

For the **logit rule,** a randomly chosen player (x) modifies her strategy from s_x to s'_x with probability

$$w(\mathbf{s}_x \to \mathbf{s}'_x) = \frac{\exp[u_x(s'_x)/K]}{\sum_{s''_x} \exp[u_x(s''_x)/K]}$$

Repeating this elementary strategy update, the system evolves into the Boltzmann distribution.

In the Boltzmann distribution, the probability of the microscopic state $\mathbf{s}=(s_1, \ldots, s_N)$:

$$p(\mathbf{s}) = \frac{1}{Z}\exp(U(\mathbf{s})/K), \text{ where } Z = \sum_{\{\mathbf{s}\}}\exp(U(\mathbf{s})/K)$$

and Z is the partition function.
In fact, the logit rule guarantees that detailed balance is satisfied for any possible (forwards and backwards) unilateral transition pairs ($s_x \leftrightarrow s'_x$) in the Boltzmann distribution, i.e.,

$$p(\mathbf{s})w(\mathbf{s}_x \to \mathbf{s}'_x) = p(\mathbf{s}')w(\mathbf{s}'_x \to \mathbf{s}_x)$$

The Boltzmann distribution can be derived as the distribution that maximizes entropy under the right constaints using the calculus of variations.

Consequences:

Thermodynamical equilibrium, phase transitions, etc.
We can use the methods of statistical physics
e.g., mean-field and pair approximations, duality relations, exact solutions, …

Derivation of the Boltzmann distribution from the principle of maximum entropy

Brief survey

Entropy: $$S = -\sum_{\{\mathbf{s}\}} p(\mathbf{s}) \ln p(\mathbf{s})$$

Constraints: normalization: $$\sum_{\mathbf{s}} p(\mathbf{s}) = 1$$

average potential: $$\sum_{\mathbf{s}} U(\mathbf{s})\, p(\mathbf{s}) = \langle U \rangle$$

According to the calculus of variations, we should solve the equations

$$\frac{\partial}{\partial p(\mathbf{s})}\left[-\sum_{\{\mathbf{s}\}} p(\mathbf{s}) \ln p(\mathbf{s}) + \alpha \sum_{\{\mathbf{s}\}} p(\mathbf{s}) + \beta \sum_{\{\mathbf{s}\}} U(\mathbf{s})\, p(\mathbf{s}) \right] = 0 \quad \text{for} \quad \forall\ p(\mathbf{s})$$

with Lagrange multiplicators α and β.

Derivations and algebraic manipulations yield:

$$\left[-1 - \ln p(\mathbf{s}) + \alpha + \beta U(\mathbf{s})\right] = 0 \quad \text{for} \quad \forall\ p(\mathbf{s})$$

$$\ln p(\mathbf{s}) = -1 + \alpha + \beta U(\mathbf{s})$$

$$p(\mathbf{s}) = e^{-1+\alpha+\beta U(\mathbf{s})} = \frac{1}{Z} e^{\beta U(\mathbf{s})}$$

In our case, $\beta = 1/K$.

We can introduce a thermodynamic potential (via Lagrange transformation) as

$$\Phi(K) = \langle U \rangle + KS = \sum_{\{\mathbf{s}\}} p(\mathbf{s}) U(\mathbf{s}) - K \sum_{\{\mathbf{s}\}} p(\mathbf{s}) \ln p(\mathbf{s}),$$

which is maximized at the Boltzmann distribution under variations of $p(\mathbf{s})$.

In mean-field approximation, translation invariant systems can be characterized by the probability $p_1(i)$ of finding the ith strategy at any site x that can be defined by $(n–1)$ positive probability parameters, because $\Sigma_i p_1(i)=1$. In the absence of correlation between neighbouring sites, this means

$$p(\mathbf{s}) = \prod_{s_x} p_1(s_x).$$

The entropy and the average potential then take the following simple forms:

$$S = -N \sum_i p_1(i) \ln(p_1(i))$$

$$\langle U \rangle = \frac{Nz}{2} \sum_{i,j} p_1(i) V_{ij} p_1(j)$$

For example, if $n=2$ then $p_1(1)=1/2+y$ and $p_1(2)=1/2–y$ and the equilibrium solution will be given by the solution of the equation

$$\frac{\partial \Phi}{\partial y} = 0$$

for which Φ is the largest.

Similar methods can be used when there are more probability parameters.

Mapping symmetric 2x2 games onto the (magnetic) Ising model

Pair interactions in social dilemma notation (*D* and *C* strategies):

$$A^{(sd)} = \begin{pmatrix} 0 & T \\ S & 1 \end{pmatrix} = \frac{1+T+S}{4}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \frac{1-T-S}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{-1+T-S}{4}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \frac{-1-T-S}{4}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$\mathbf{V}^{(sd)} = \frac{1-T-S}{4}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{-1+T-S}{4}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

Magnetic Ising model: σ_x=+1, –1 (up and down spin states)

the negative Hamiltonian:

$$-H = J\sum_{\langle x,y\rangle}\sigma_x\sigma_y + h\sum_x \sigma_x = -\sum_{\langle x,y\rangle} v^{(\text{Ising})}$$

$$-v^{(\text{Ising})} = J\sigma_x\sigma_y + \frac{h}{z}(\sigma_x + \sigma_y)$$

In game theory's notation:

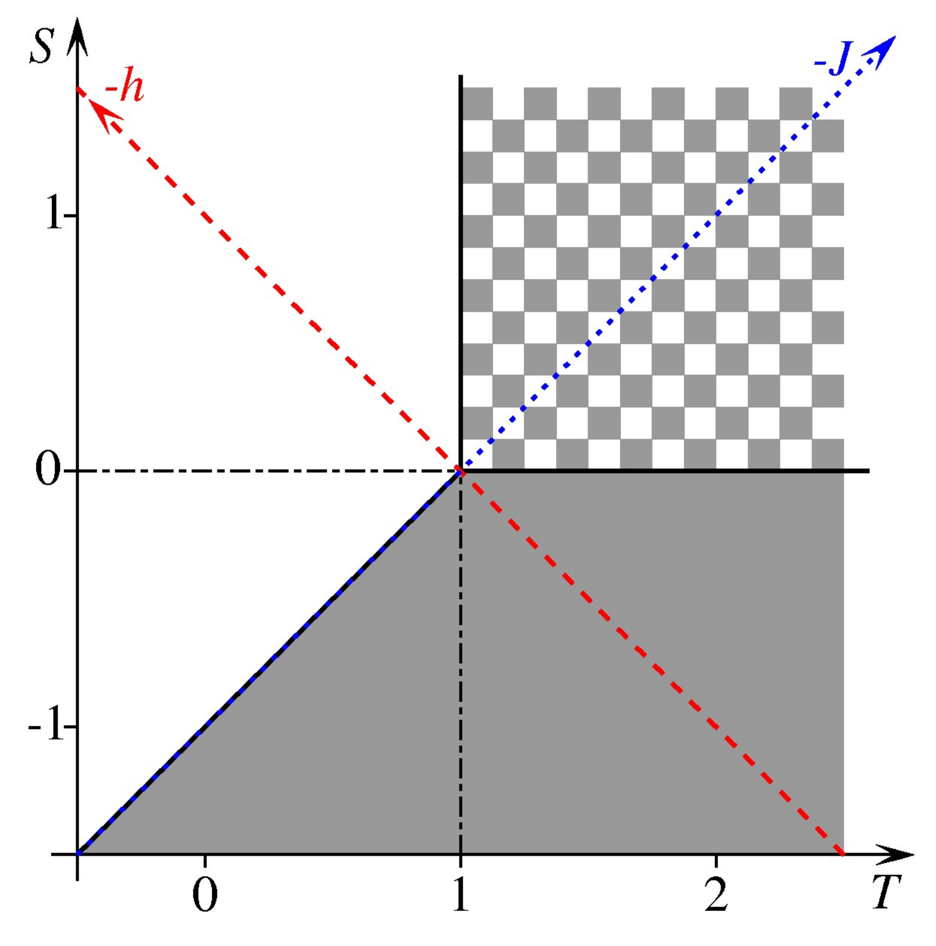
$$\sigma_x = +1 \rightarrow \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \sigma_x = -1 \rightarrow \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad \text{and} \quad -\mathbf{V}_{\text{Ising}} = \begin{pmatrix} J + \frac{2h}{z} & -J \\ -J & J - \frac{2h}{z} \end{pmatrix} = J\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \frac{h}{z}\begin{pmatrix} 2 & 0 \\ 0 & -2 \end{pmatrix}$$

Mapping the Ising model onto symmetric 2x2 games (cont.)

Comparing the parameters yields:

$$4J = 1 - T - S, \quad \text{and} \quad 4h/z = -1 + T - S$$

On the *S*–*T* plane of social dilemmas, the axes of *J* and *h* correspond to the red and blue dashed lines.



J>0 : ferromagnetic order

J<0 : antiferromagnetic (chequerboard) order

h>0 : spin up favoured

h<0 : spin down favoured

Phenomena in the square-lattice Ising model (z=4, and K=T: temperature)

Ferromagnetic case = Coordination game (Linux or Windows)

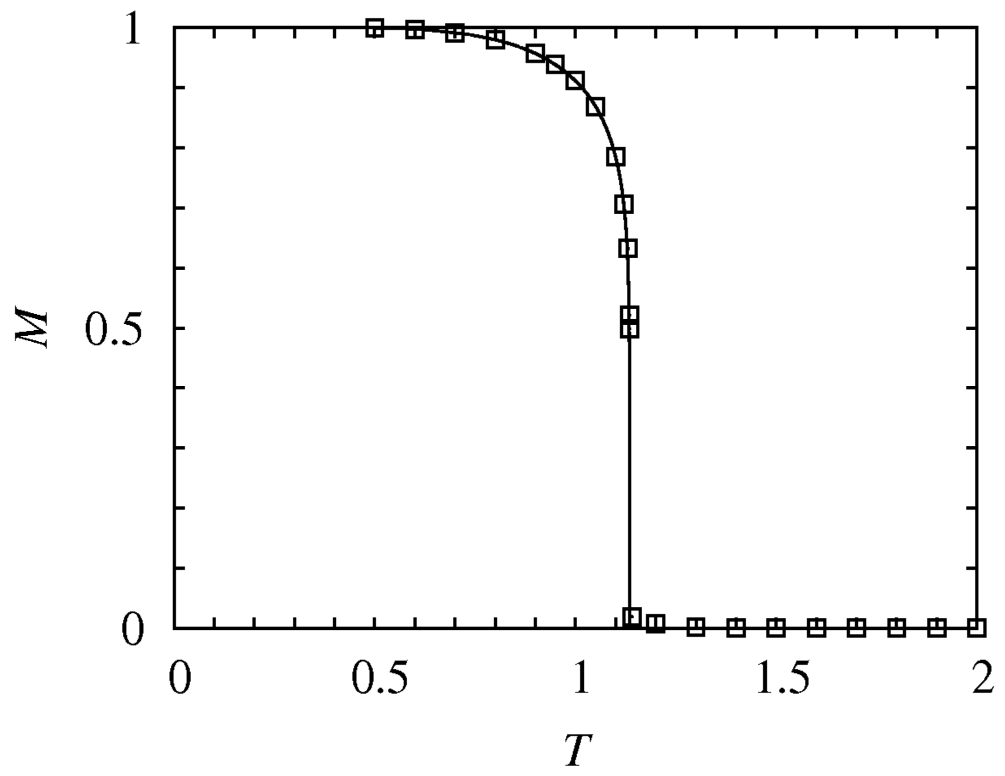
The stationary state is quantified by the average magnetization M, considered as an order parameter, varying from 0 to 1, if σ_x=1, –1:

$$M = \left|\langle \sigma_x \rangle\right|, \quad \text{where} \quad \langle \sigma_x \rangle = \frac{1}{N}\sum_x \sigma_x$$

Simulations:

ordered state

disordered state



□: Monte Carlo simulations

-----: Onsager (1944)

two-dimensional exact solution

$$M = \left\{1 - \left[\mathrm{sh}(1/T)\right]^{-4}\right\}^{1/8}$$

$$T_c = 1/\ln\left(1+\sqrt{2}\right) \approx 1.134592$$

Sublattice ordering in antiferromagnetic models

or in anticoordination games, e.g.: hawk–dove

Simulation below T_c Simulation above T_c Frozen state (small T)

The lattice can be divided into two sublattices (a and b) on the analogy of the chessboard.

The long-range ordered state is characterized by sublattice magnetizations (the difference of the magnetizations on the sublattices) that is considered as an order parameter:

$$M = |\rho_a - \rho_b|, \quad \text{where}$$

$$\rho_a = \frac{2}{N}\sum_{i\in a}\sigma_i, \quad \text{and} \quad \rho_b = \frac{2}{N}\sum_{i\in b}\sigma_i.$$

On the square lattice, the ferromagnetic model can be mapped onto an antiferromagnetic system by the following transformations:

$$\mathbf{A} \to -\mathbf{A} \quad \text{or} \quad \sigma_i \to -\sigma_i \;\; \forall i \in a$$

Consequence:

The ferromagnetic and the antiferromagnetic systems exhibit similar behaviour.

More complex behaviour if second-neighbour interactions are also allowed: sim

Relatives of the Ising model

Ising model: $\sigma_x = -1, +1$ (spin up, spin down) states

Energy:

$$H = -\frac{1}{2} J \sum_{x,\delta} \sigma_x \sigma_{x+\delta} - h \sum_x \sigma_x$$

J: coupling constant ($J > 0$: ferromagnetic)

h: external magnetic field

Lattice gas models: n_x=0, 1 (site x is empty or accupied by an atom)

$$H = -\frac{1}{2} J \sum_{x,\delta} n_x n_{x+\delta} - \mu \sum_x n_x$$

J: attractive or repulsive interaction

μ: binding energy or chemical potential

Applications: adatoms on a single crystal surface

metal–hydrogen systems

intercalation compounds

two-component alloys (CuAu, CuNi, etc.)

superionic conductors (AgI, CaF_2, ZrO_2, etc)

phase diagrams are calculated for materials science

Relatives of the Ising model (cont.)

Potts model

n equivalent states at the sites x of a lattice. Interactions exist between the neighbours.
The pair interaction potential (in matrix notation)

$$\mathbf{V} = \begin{pmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{pmatrix} \quad \text{or} \quad V_{ij} = \delta_{ij} \qquad \text{(Kronecker's delta)}$$

favours the formation of any one of n homogeneous states.

- This interaction can be considered as a combination of all possible pair coordinations with equal strength.
- Order–disorder phase transitions represent universality classes.
- simulations for n=3 and attractive interactions

Attr. Potts n=3

- simulation for n=3 and repulsive interactions

  no ordered states, no phase transitions

  the ground state is infinitely degenerate

  frustration can occur

Rep. Potts n=3

Relatives of the Ising model (cont.)

New opportunities for finding equivalent ferro- and antiferromagnetic systems

Example for n=3

The pair interaction/potential (in matrix notation)

$$\mathbf{V} = \begin{pmatrix} 2 & -1 & -1 \\ -1 & -1 & 2 \\ -1 & 2 & -1 \end{pmatrix}$$

- three equivalent pure Nash equilibria
- simulations on a square lattice

curiousity n=3

- exchanging labels 2 and 3 on one of the sublattices transforms the system into the three-state Potts model

Ordering process

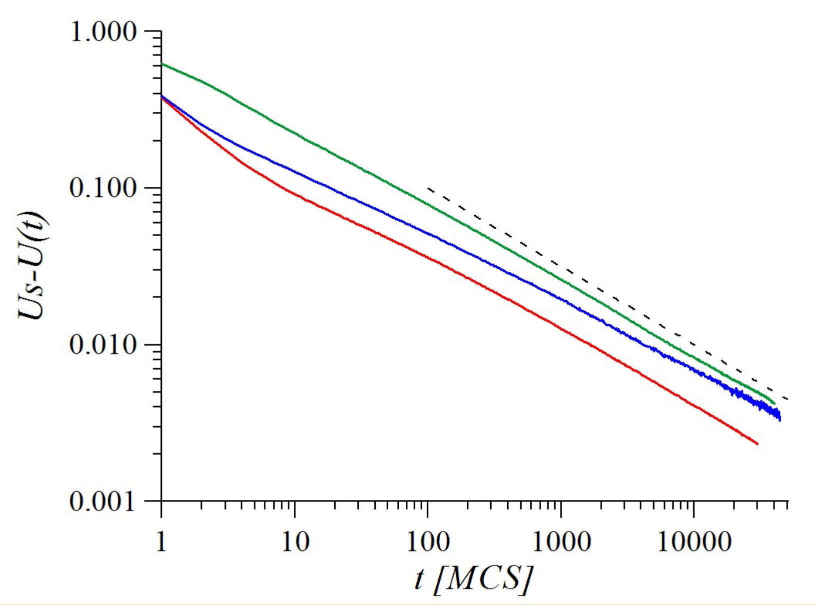
The typical domain size increases with time as $t^{1/2}$, if the system is started from a random initial state below the critical noise level (temperature) ($T<T_c$)

Simulations for several temperatures

U_s : Potential (energy) in the stationary state (t=∞)

$U(t)$: time-dependent potential

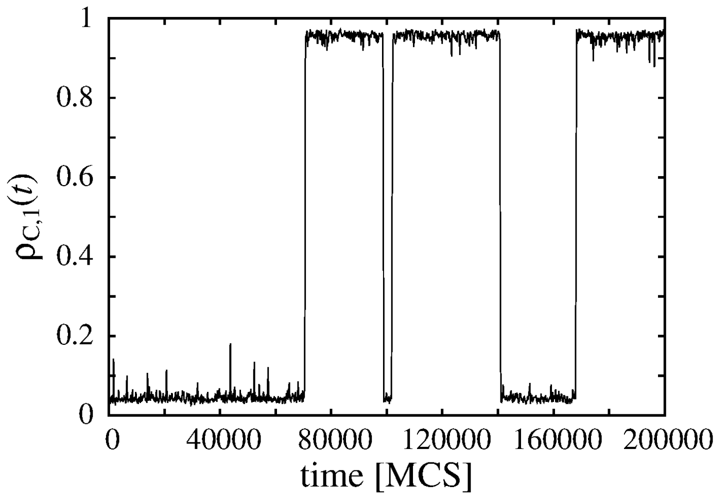
Dashed line : $t^{-1/2}$



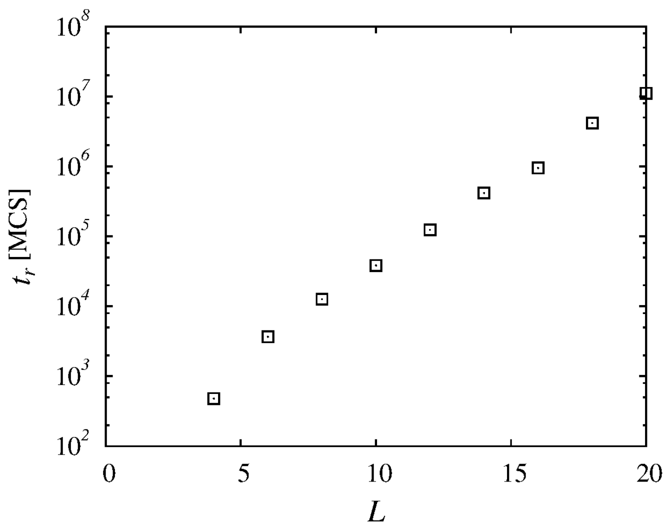
Ergodicity:

ensemble average and time average are different.

C frequency vs. *t* in sublattice 1 for *L*=16.



Avg. reversal time versus size *L*



Home assignment

8.1. Show that mean-field approximation predicts a first-order phase transition for the four-state Potts model on a square lattice!