Nash equilibria in symmetric 2x2 games:

strategies in social dilemma notation:

D: defection (black bullet)

C: cooperation (white bullet)

Four regions:

1.) **Harmony games** (H): $T<1$, $S>0$

one pure Nash equilibrium: *CC* (no problems)

2.) **Hawk–dove games** (HD): $T>1$, $S>0$

two equivalent pure Nash equilibria: *CD* and *DC*

3.) **Stag hunt games** (SH): $T<1$, $S<0$

preferred Nash equilibrium: *DD* if $S-T+1<0$

CC if $S-T+1>0$

4.) **Prisoner's dilemmas** (PD): $T>1$, $S<0$

one preferred Nash equilibrium: *DD*

There are only four different flow graphs!

Payoff matrix:

$$\mathbf{A} = \begin{pmatrix} P & T \\ S & R \end{pmatrix}$$

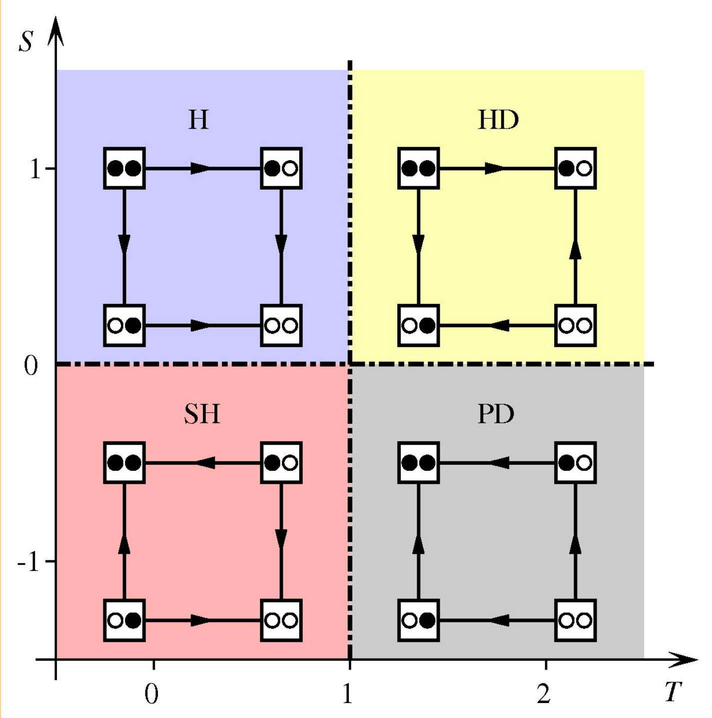
R: Reward for mutual cooperation

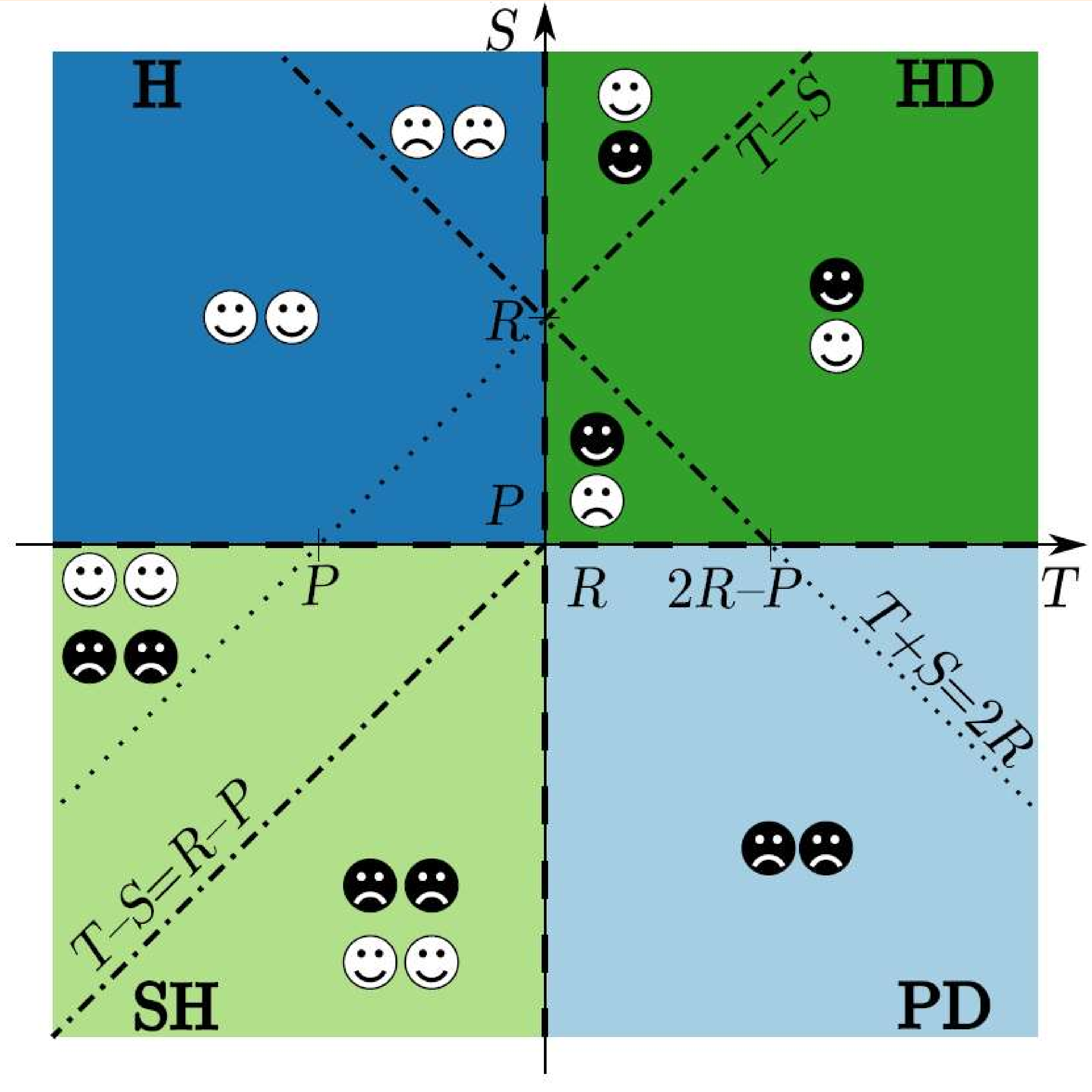
P: Punishment for mutual defection

T: Temptation to choose defection

S: Sucker's payoff

Simplification: $P=0$ and $R=1$





S
H
HD
T=S
R
P
T
P
R
2R–P
T+S=2R
T–S=R–P
SH
PD

Additional stability criteria

for populations playing repeated games with Darwinian selection

n strategies (species), N players ($N \to \infty$)

Strategy i (i=1, …, n) is followed by N_i players.

Frequency vector: $\boldsymbol{\rho}=(\rho_1, \rho_2, \ldots, \rho_n)$, where

$$\rho_i = \frac{N_i}{N}, \qquad \sum_i \rho_i = 1; \quad \text{simplex} \quad \boldsymbol{\rho} \in \Delta_n$$

This population dynamics is similar to mean-field approximation in physics.

The payoff **A** we identify as evolutionary fitness, the capability to create offspring, resulting from interactions between members of the species.

$\boldsymbol{\rho}^*$ is an **Evolutionarily Stable Strategy** (ESS), if no invading mutant $\boldsymbol{\rho}' \neq \boldsymbol{\rho}^*$ can outcompete it, that is, if its fitness is lower in small admixtures:

$$\boldsymbol{\rho} = (1-\varepsilon)\boldsymbol{\rho}^* + \varepsilon\boldsymbol{\rho}' \qquad \boldsymbol{\rho}^*, \boldsymbol{\rho}' \in \Delta_n$$

small perturbation in the population, $\varepsilon \to 0$ limit

$$\boldsymbol{\rho}^* \mathbf{A} \boldsymbol{\rho} > \boldsymbol{\rho}' \mathbf{A} \boldsymbol{\rho}$$

when equality is allowed: weak ESS

This concept considers stability without considering dynamical details.

The concept of ESS was introduced by Maynard Smith and Price in *Nature* 246 (1973) 15.

After plugging in the definition of $\boldsymbol{\rho}$ and rearranging the resulting expression:

$$(1-\varepsilon)\boldsymbol{\rho}^{*}\mathbf{A}\boldsymbol{\rho}^{*}+\varepsilon\boldsymbol{\rho}^{*}\mathbf{A}\boldsymbol{\rho}'>(1-\varepsilon)\boldsymbol{\rho}'\mathbf{A}\boldsymbol{\rho}^{*}+\varepsilon\boldsymbol{\rho}'\mathbf{A}\boldsymbol{\rho}'$$

This is satisfied for arbitrarily small ε, if

1.) $\boldsymbol{\rho}^{*}\mathbf{A}\boldsymbol{\rho}^{*}\geq\boldsymbol{\rho}'\mathbf{A}\boldsymbol{\rho}^{*}$ and

2.) if $\boldsymbol{\rho}^{*}\mathbf{A}\boldsymbol{\rho}^{*}=\boldsymbol{\rho}'\mathbf{A}\boldsymbol{\rho}^{*}$, then $\boldsymbol{\rho}^{*}\mathbf{A}\boldsymbol{\rho}'>\boldsymbol{\rho}'\mathbf{A}\boldsymbol{\rho}'$

That is, if the fitness of $\boldsymbol{\rho}^*$ is at least as large as the fitness of mutants in its own population and exceeds it in the mutant populations.

If $\boldsymbol{\rho}^*$ is an ESS, then $(\boldsymbol{\rho}^*, \boldsymbol{\rho}^*)$ is a symmetric NE, which is isolated from other symmetric NEs.

If $(\boldsymbol{\rho}^*, \boldsymbol{\rho}^*)$ is a strict NE, then $\boldsymbol{\rho}^*$ is an ESS.

There can exist multiple ESSs, and there are systems without any ESSs.

Determination of ESS in the hawk–dove game

$$\begin{array}{ccc} & \text{hawk} & \text{dove} \end{array}$$
$$\mathbf{A} = \begin{array}{c}\text{hawk}\\ \text{dove}\end{array}\begin{pmatrix} (v-c)/2 & v \\ 0 & v/2 \end{pmatrix} \quad , \quad c > v > 0$$

Three NE:

$$s_{x1}^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, s_{y1}^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad s_{x2}^* = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, s_{y1}^* = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad s_{x3}^* = s_{y3}^* = \begin{pmatrix} v/c \\ 1-v/c \end{pmatrix}.$$

Let us consider the evolutionary stability of the mixed NE

$$\boldsymbol{\rho}^* = \begin{pmatrix} v/c \\ 1-v/c \end{pmatrix} \quad \text{against} \quad \boldsymbol{\rho} = \begin{pmatrix} \rho \\ 1-\rho \end{pmatrix}$$

Payoff of the second player choosing $\boldsymbol{\rho}$ while her coplayer follows the mixed NE:

$$\boldsymbol{\rho}\mathbf{A}\boldsymbol{\rho}^* = \rho\frac{v}{c}\frac{v-c}{2} + \rho\left(1-\frac{v}{c}\right)v + (1-\rho)\left(1-\frac{v}{c}\right)\frac{v}{2}$$

$$= \frac{1}{2}v - \frac{1}{2}\frac{v^2}{c}$$ independently of ρ, so $(\boldsymbol{\rho}^*, \boldsymbol{\rho}^*)$ is a (weak) symmetric NE

The second criterion of ESS:

$$\boldsymbol{\rho}^{*}\mathbf{A}\boldsymbol{\rho} > \boldsymbol{\rho}\mathbf{A}\boldsymbol{\rho}$$

$$\boldsymbol{\rho}^{*}\cdot\mathbf{A}\boldsymbol{\rho} = \rho\frac{v(v-c)}{2c} + (1-\rho)\frac{v}{c}v + (1-\rho)\left(1-\frac{v}{c}\right)\frac{v}{2}$$

$$\boldsymbol{\rho}\cdot\mathbf{A}\boldsymbol{\rho} = \rho^{2}\frac{(v-c)}{2} + (1-\rho)\rho v + (1-\rho)(1-\rho)\frac{v}{2}$$

$$\begin{aligned}\boldsymbol{\rho}^{*}\mathbf{A}\boldsymbol{\rho} - \boldsymbol{\rho}\mathbf{A}\boldsymbol{\rho} &= \rho\left(\frac{v}{c}-\rho\right)\frac{v-c}{2} + (1-\rho)\left(\frac{v}{c}-\rho\right)v - (1-\rho)\left(\frac{v}{c}-\rho\right)\frac{v}{2} \\ &= \left(\frac{v}{c}-\rho\right)\left(\rho\frac{v-c}{2} + (1-\rho)\frac{v}{2}\right) = \frac{c}{2}\left(\frac{v}{c}-\rho\right)^{2} > 0,\end{aligned}$$

that is, $\boldsymbol{\rho}^{*}\mathbf{A}\boldsymbol{\rho} > \boldsymbol{\rho}\mathbf{A}\boldsymbol{\rho}$,

Thus $\boldsymbol{\rho}^{*}$ is an ESS.

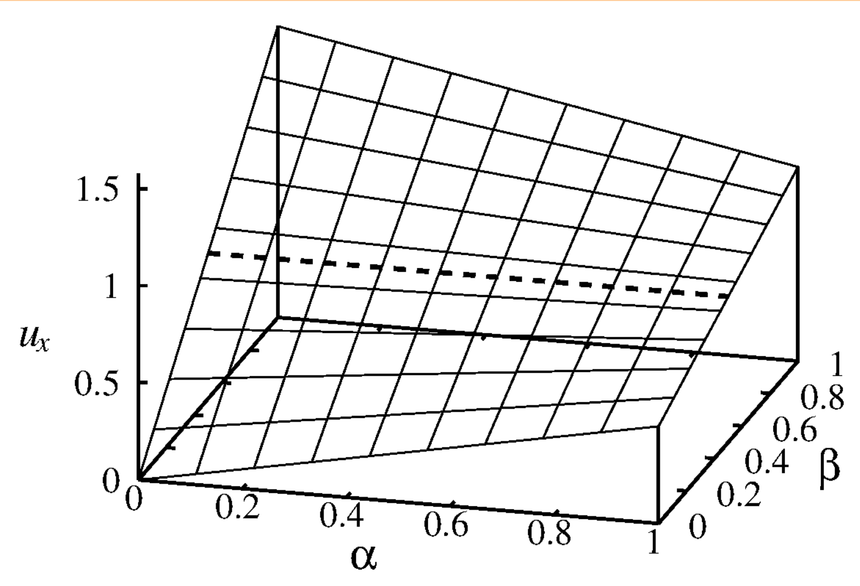
Payoffs for symmetric two-player, two-strategy games for mixed strategies:

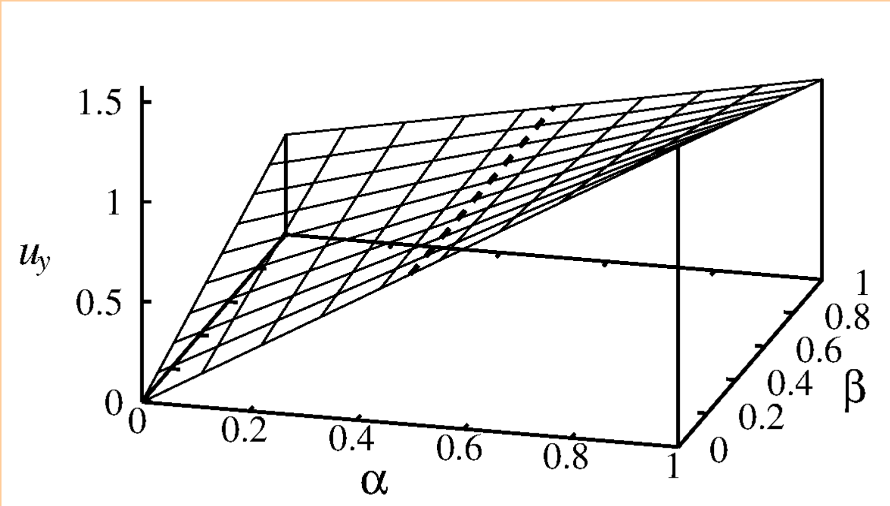
Social dilemma notation:

$$\mathbf{A} = \begin{pmatrix} (0,0) & (T,S) \\ (S,T) & (1,1) \end{pmatrix}$$

Two mixed strategies: $s_x = \begin{pmatrix} 1-\alpha \\ \alpha \end{pmatrix}, \quad s_y = \begin{pmatrix} 1-\beta \\ \beta \end{pmatrix}$

Payoffs (u_x and u_y) for player x and y as a function of α and β, if T=1.5 and S=0.5 (HD game)





The mixed NE: α^*=0.5 and β^*=0.5

Notice: if one of the players chooses the mixed NE then the other's payoff is constant (see dashed lines)

Replicator dynamics Taylor and Jonker, *Math. Biosci*. 40 (1978) 145

In biology: strategy = species and payoff = fitness (offspring creation capability)

N strategies (species): ρ_i (i=1, …, N) is the concentration of strategy i in the population.

According to Darwin's rule of selection: $\rho_i(t)$ depends on time and successful species increase their proportion in the population at the expense of unsuccessful ones.

Taylor and Jonker's formula:

$$\frac{\dot{\rho}_i}{\rho_i} = (\text{fitness})_i - (\text{average fitness}), \qquad \text{that is,}$$

$$\dot{\rho}_i = \rho_i[(\mathbf{A}\boldsymbol{\rho})_i - \boldsymbol{\rho}\cdot\mathbf{A}\boldsymbol{\rho}]$$

Maynard Smith and Price's formula (difference in time scale)

$$\dot{\rho}_i = \rho_i \frac{(\mathbf{A}\boldsymbol{\rho})_i - \boldsymbol{\rho}\cdot\mathbf{A}\boldsymbol{\rho}}{\boldsymbol{\rho}\cdot\mathbf{A}\boldsymbol{\rho}} \qquad \left[\boldsymbol{\rho}\cdot\mathbf{A}\boldsymbol{\rho} > 0 \text{ is assumed}\right]$$

The stationary states (fixed points) are the same.

Each homogeneous state (i.e., when only one species exists) **is a stationary state** as there is no payoff difference for that one species and all other ρ_i remain zero.

Classification of stationary states

Stationary solution (fixed point) $\boldsymbol{\rho}^*$: $\dot{\rho}_i = 0 \quad \forall i$

$\boldsymbol{\rho}^*$ is stable, if for any (small) open neighbourhood U of $\boldsymbol{\rho}^*$ exists a neighbourhood

$$O \subseteq U \quad \text{such that} \quad \boldsymbol{\rho}(t \geq 0) \subseteq U \quad \text{if} \quad \boldsymbol{\rho}(t = 0) \in O$$

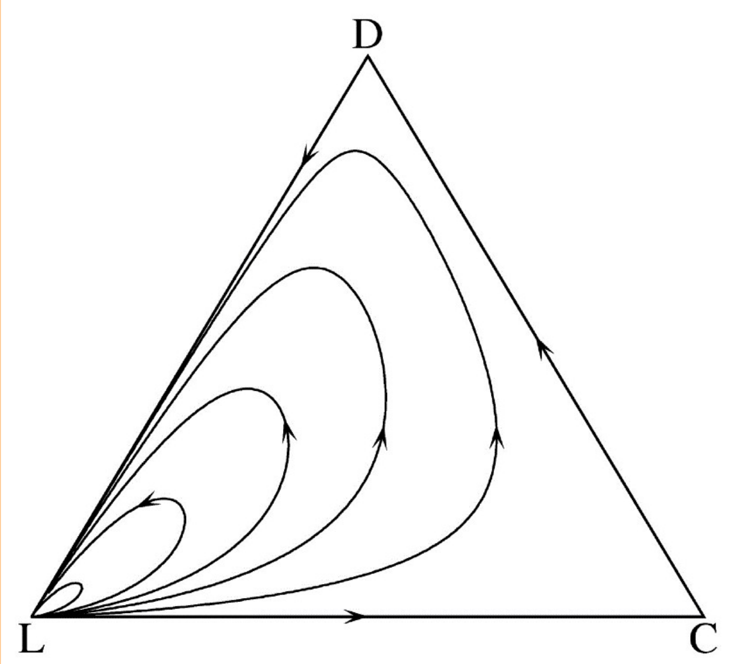
$\boldsymbol{\rho}^*$ is unstable, if it is not stable

$\boldsymbol{\rho}^*$ is attractive, if it has an open neighborhood U such that $\lim_{t\to\infty} \boldsymbol{\rho}(t) = \boldsymbol{\rho}^*$, if $\boldsymbol{\rho}(t = 0) \in U$

Basin of attraction: largest such U

$\boldsymbol{\rho}^*$ is asymptotically stable (an attractor) if it is both stable and attractive.

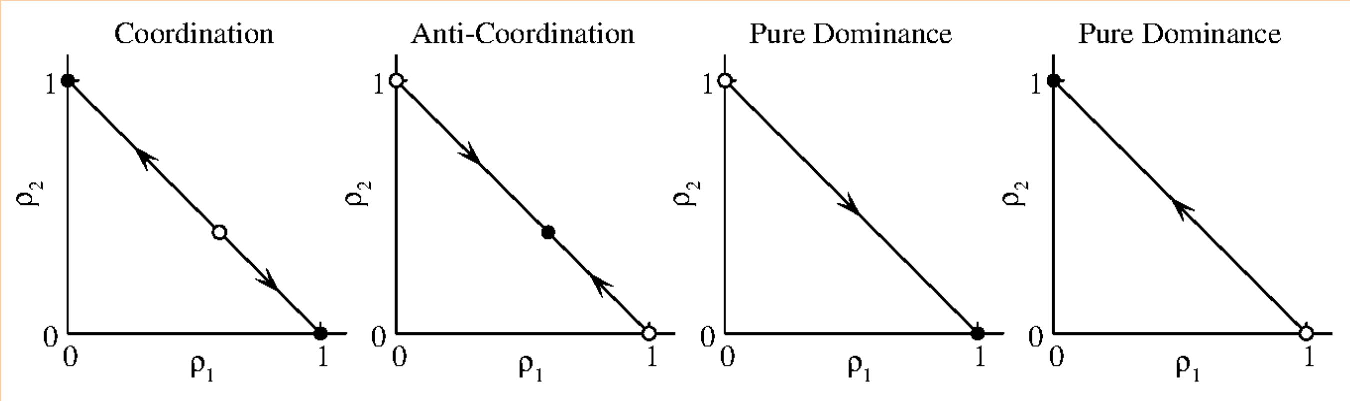
Unstable fixed point may be attractive, see L in this simplex.



Replicator dynamics

SH=stag hunt, HD=hawk–dove, PD=prisoner's dilemma, H=harmony

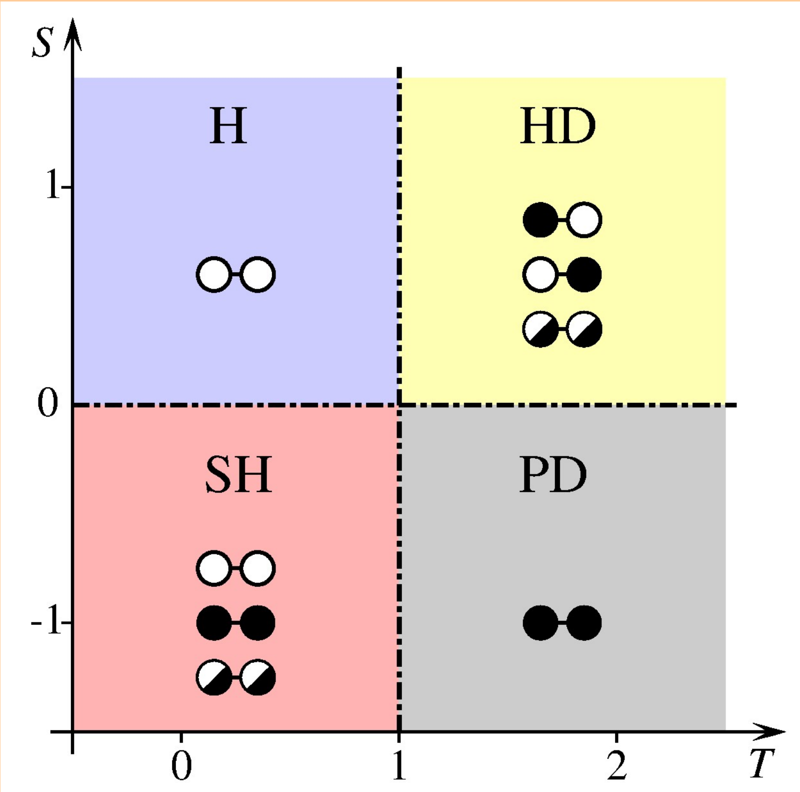
Typical solutions for n=2:



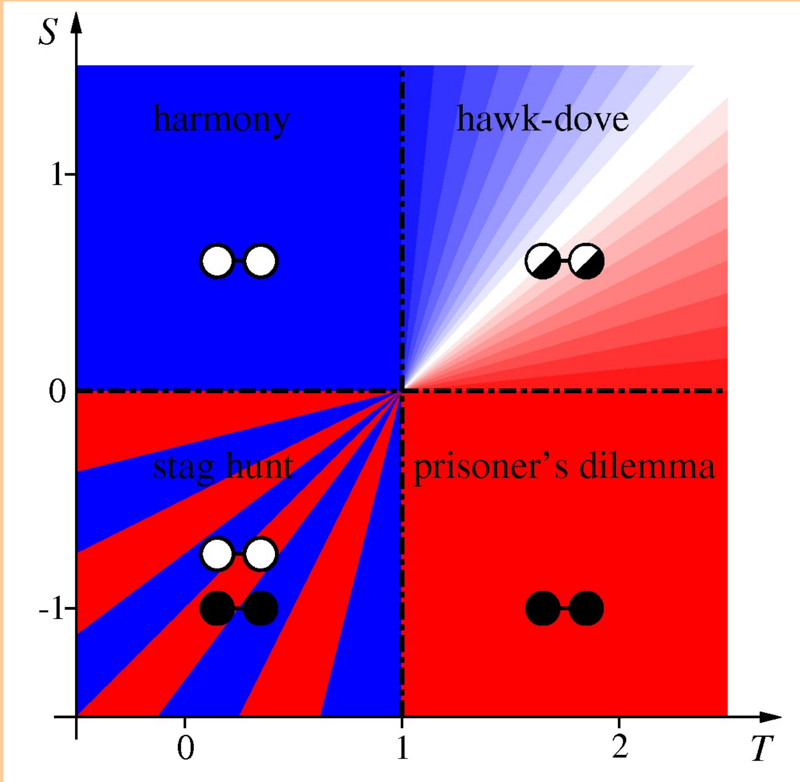
Social dilemmas

$$\mathbf{A} = \begin{pmatrix} (0,0) & (T,S) \\ (S,T) & (1,1) \end{pmatrix}$$

Nash equilibria

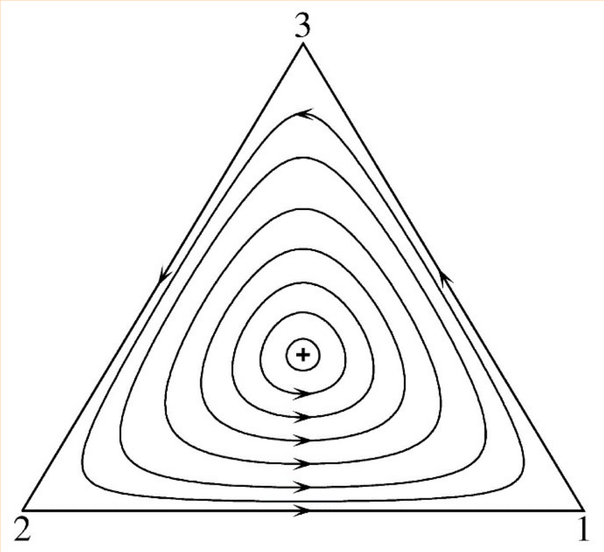


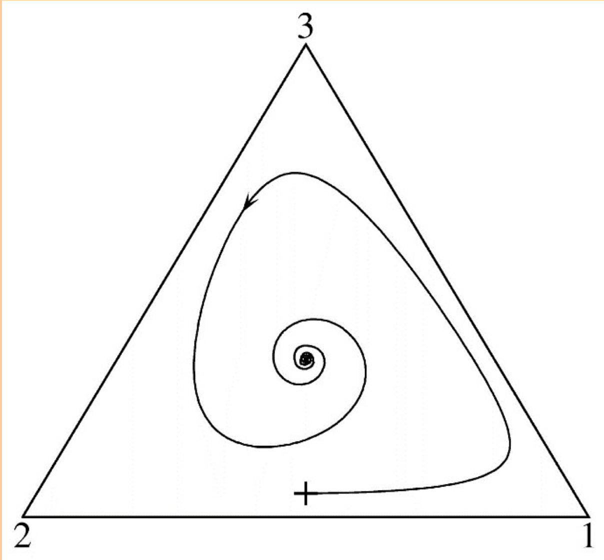
Solutions for population dynamics

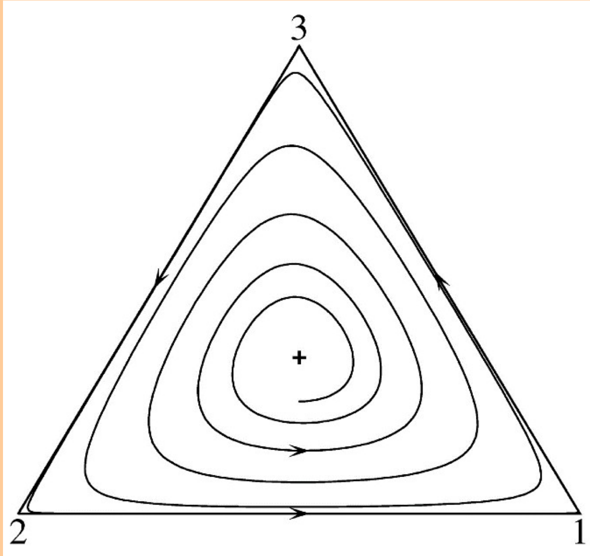


Replicator dynamics

Rock–paper–scissors model (n=3) in the presence of coordination:







Concentric trajectories spiral in spiral out

Common features

Four stationary solutions: 1-3) $\rho_1=1$; $\rho_2=\rho_3=0$, etc. (homogeneous, unstable)

4) $\rho_1=\rho_2=\rho_3=1/3$ (stable or unstable)

The complete classification of all solutions is prevented by the large number of possibilities.

Chaotic solutions can also occur for $n>3$.

Comparison of the stability criteria

Dynamical stability vs. NE

a.) NE are fixed points

b.) Strict NE are attractors.

(But not all attractors are strict NE.)

c.) If an internal trajectory converges to $\boldsymbol{\rho}^*$,

then $\boldsymbol{\rho}^*$ is a NE.

d.) Stable fixed points are NE.

Dynamical stability vs. ESS

a.) ESSs are attractors.

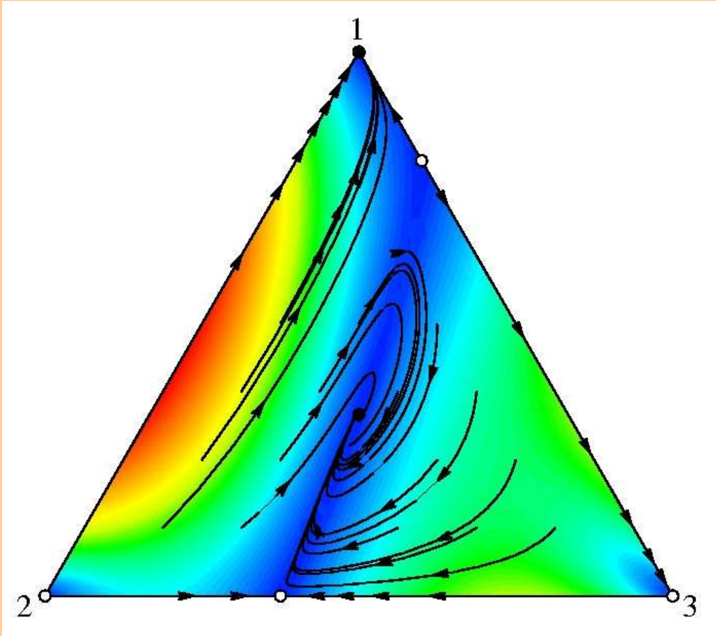
b.) Internal ESS is a global attractor.

Example with many NEs and fixed points

strict NE

ESS

↓

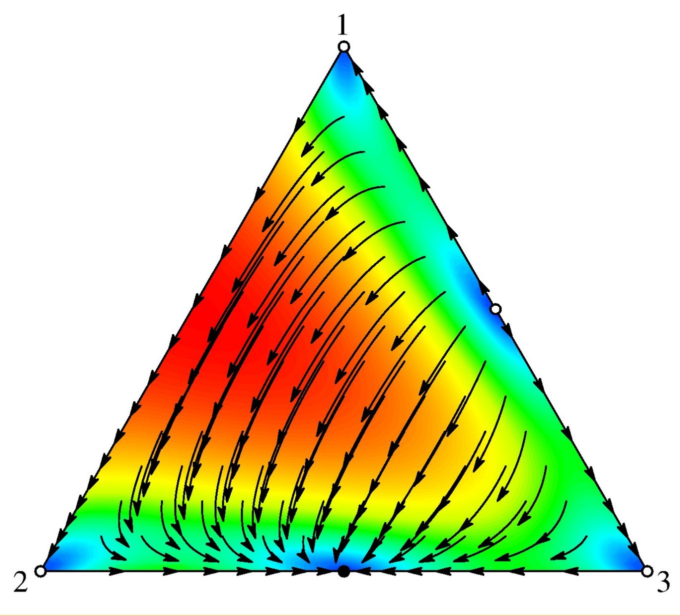


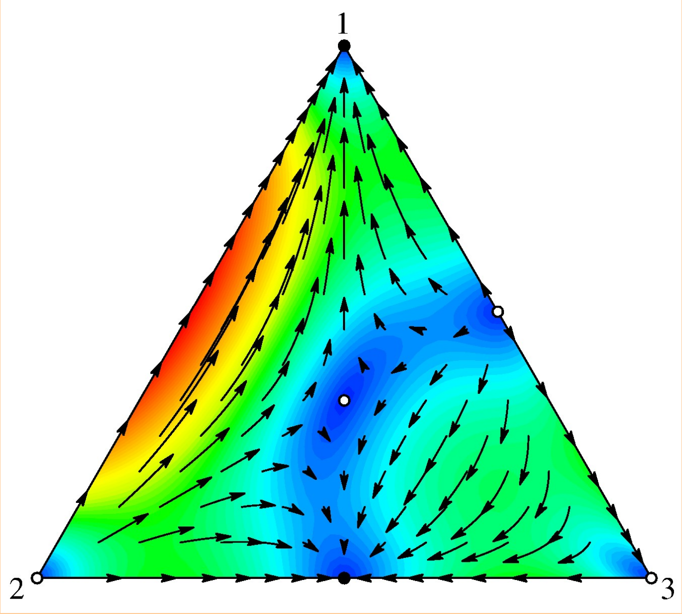
c.) For potential games, a fixed point is an ESS if and only if it is an attractor.

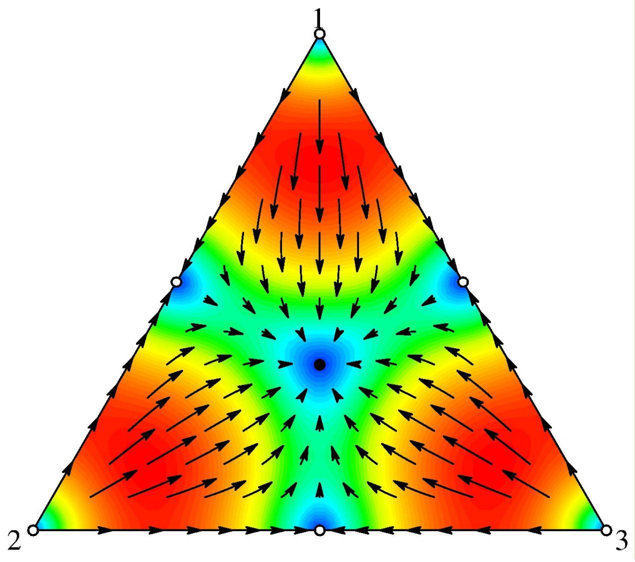
d.) For 2x2 matrix games, a fixed point is an ESS if and only if it is an attractor

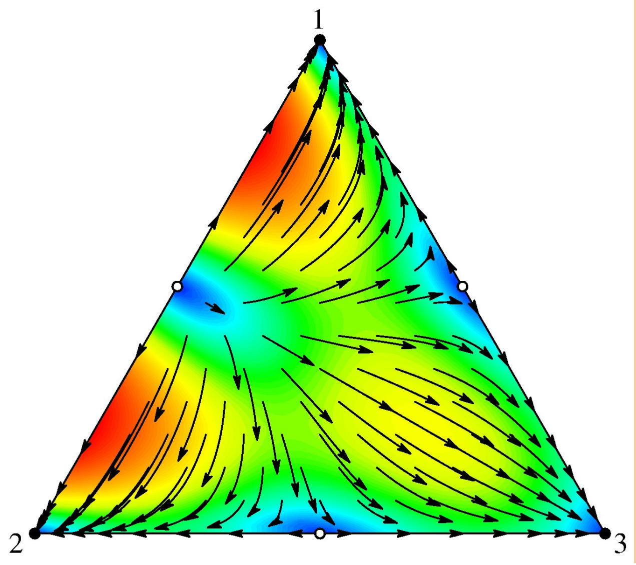
Further examples (n=3) [figures made with Dynamo (Sandholm and Dokumaci, 2006)]

Black bullets: stable solutions (attractors); White bullets: unstable solutions

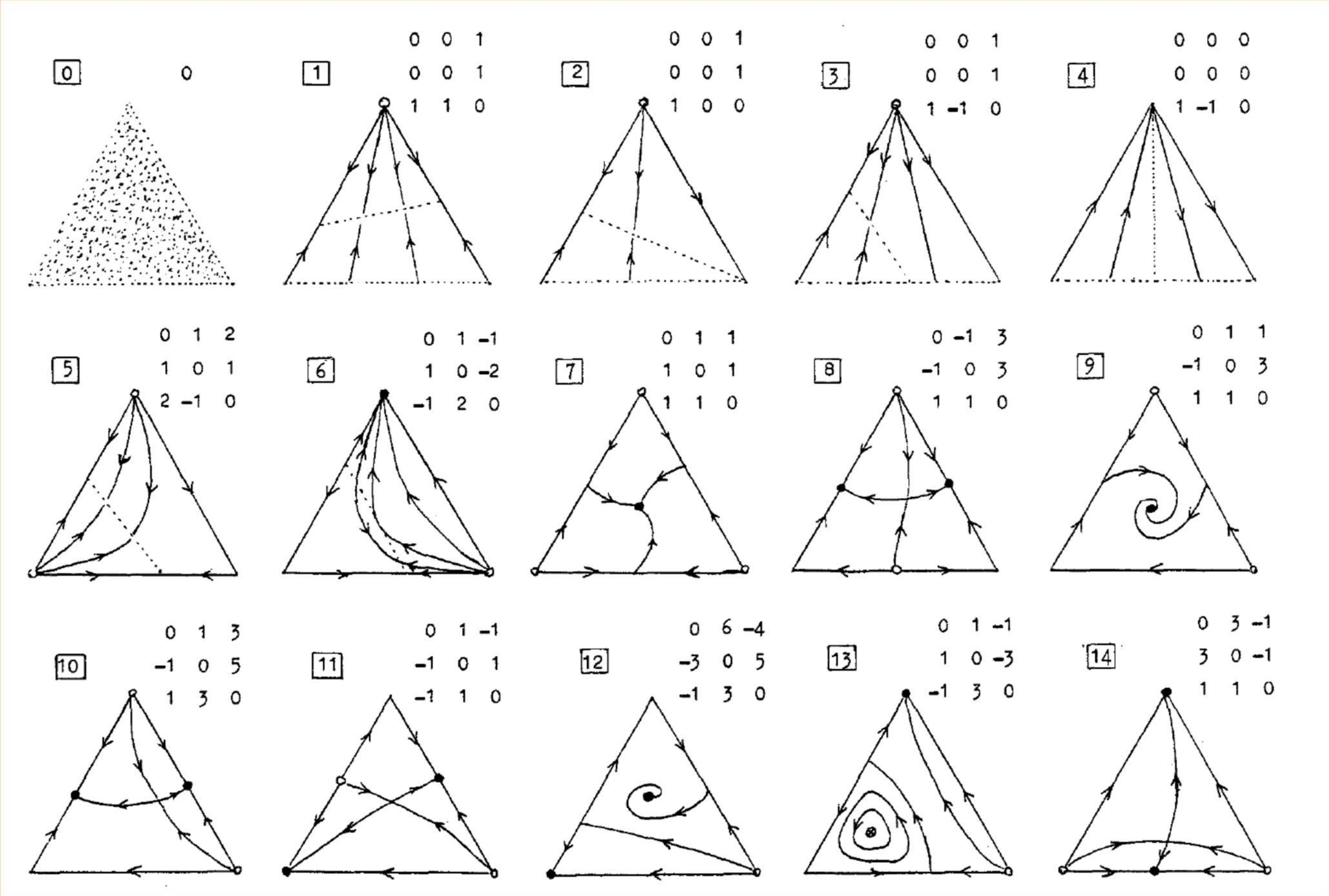


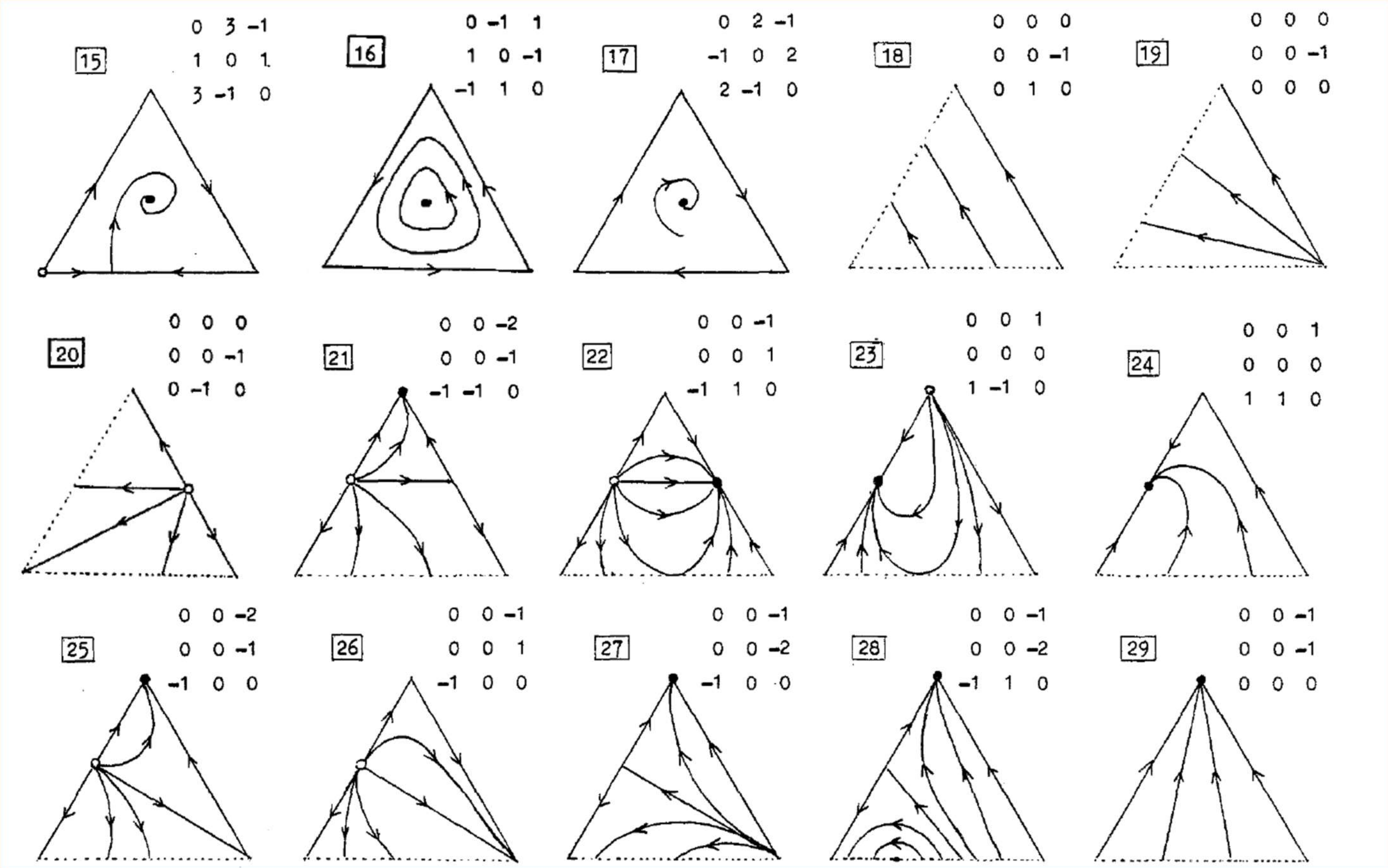






The complete *n*=3 zoo (according to Bomze, *Biol. Cybern.* 48 (1983) 201,
and Bomze, *Biol. Cybern.* 72 (1995) 447)





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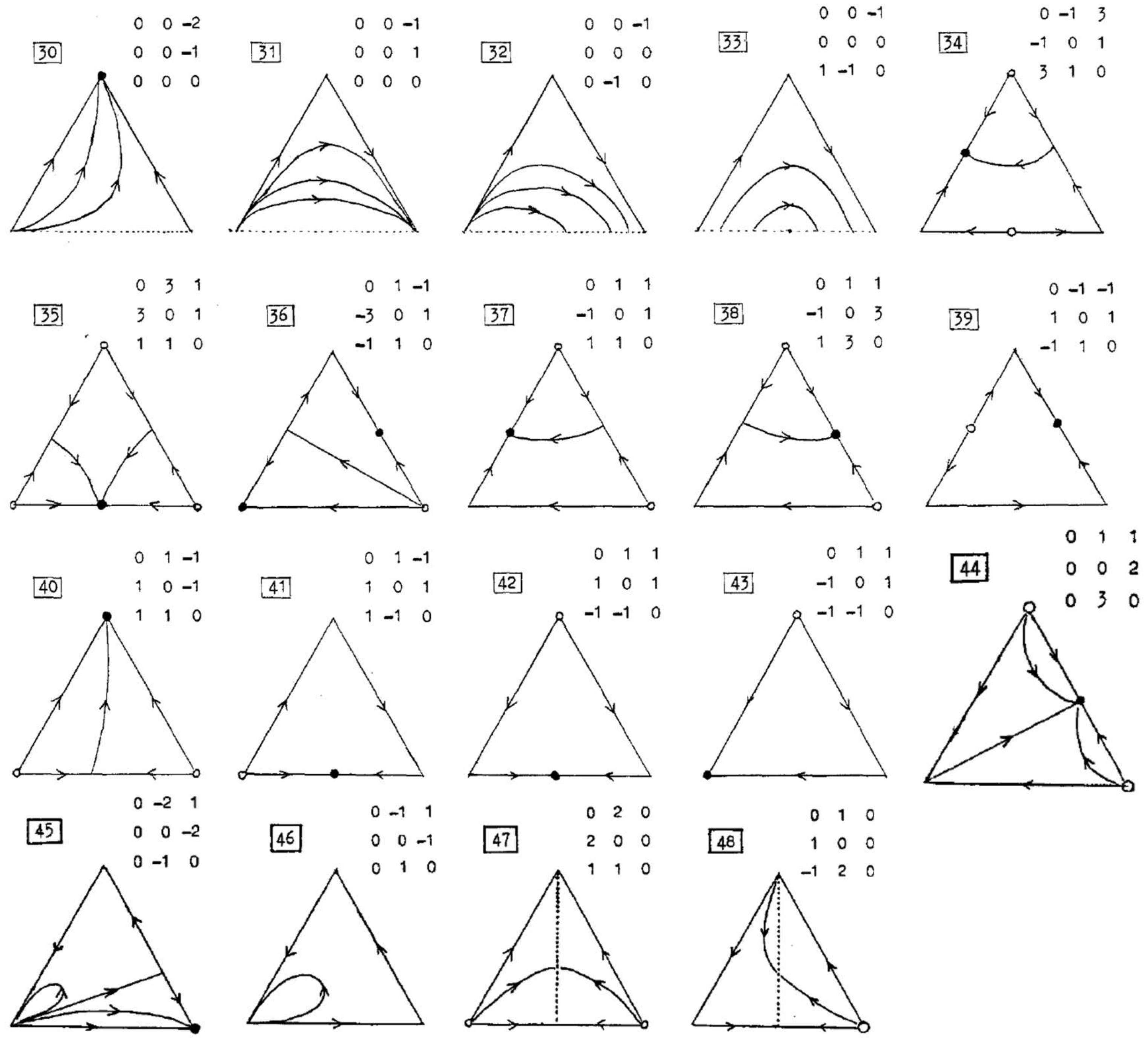
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