Spatial social dilemmas with three strategies **Lecture 12**

Example: voluntary weak prisoner's dilemma on the square lattice

Strategies:

$$s_x = D = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \quad \text{(defector)}, \quad C = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{(cooperator)}, \quad L = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \text{(loner)}$$

Payoff matrix:

$$\mathbf{A} = \begin{pmatrix} 0 & b & \sigma \\ 0 & 1 & \sigma \\ \sigma & \sigma & \sigma \end{pmatrix}, \quad 1 < b < 2, \quad 0 < \sigma < 1$$

The strategies cyclically dominate each other:

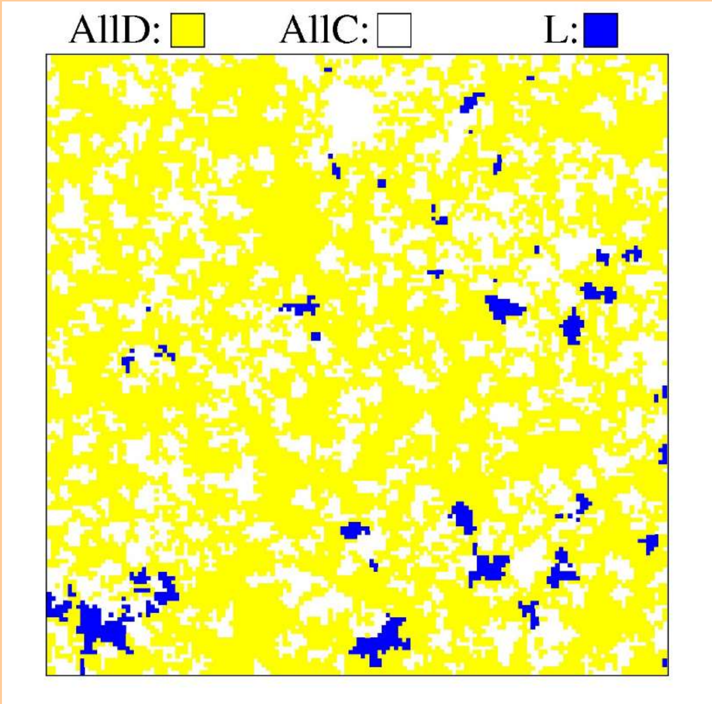
C beats L

L beats D

D beats C

Consequence: self-organizing pattern

Simulation on a 2D lattice



Effect of connectivity structure

Four connectivity structures: (z=4)

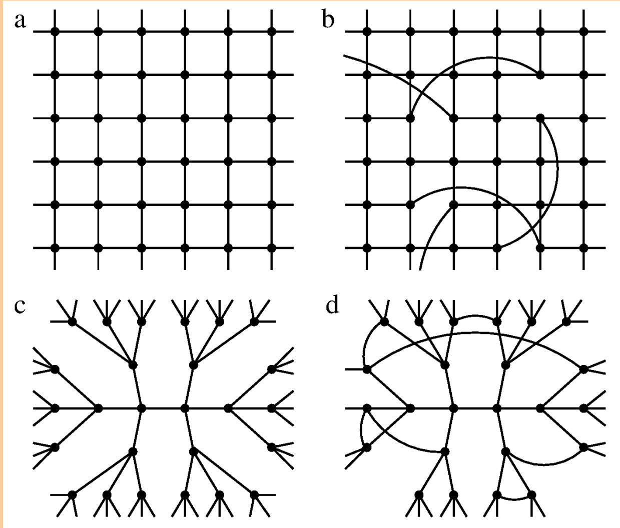
MC simulations vs b,

if K=0.1, σ=0.3, N=10^6

Frequencies of C(◊), D(□) and L(Δ) on SL

Lines: pair approximation

stable (solid) and unstable (dashed) sol.



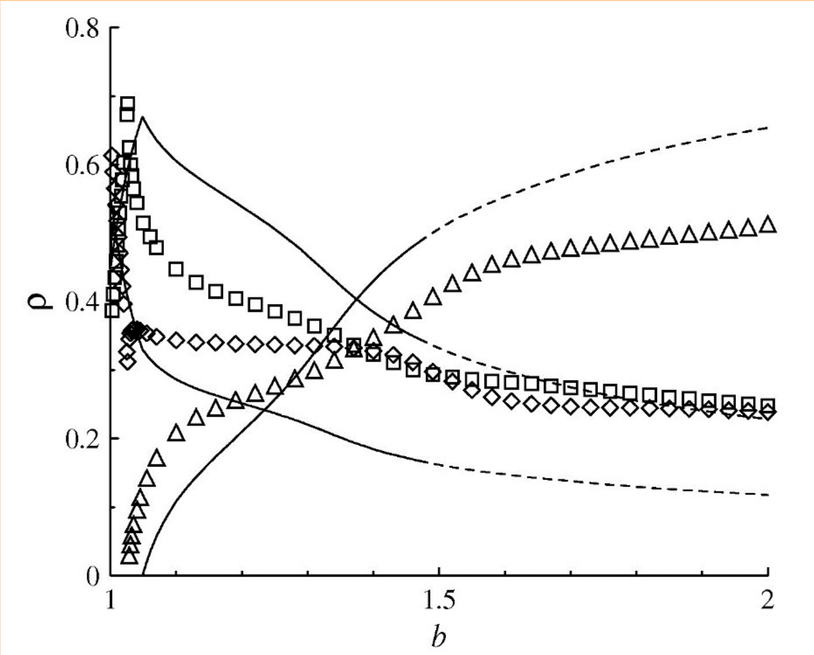
a) Square lattice

b) Newman–Watts graph

c) Bethe lattice

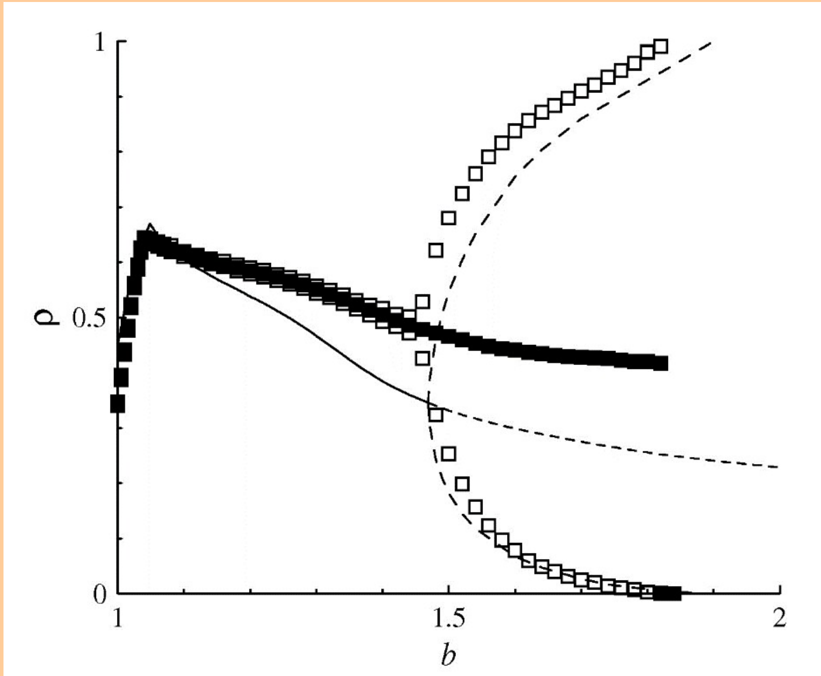
d) Random regular graphs

Locally similar structures



Notice a discrepancy: Increasing b provides higher income for D, but it's the the population of L that increases

Frequency of D on random regular graph



Global oscillation on SL with 0.02 portion of RL

Evolutionary rock–paper–scissors games on graphs

Three strategies:

rock crushes scissors

scissors cut paper

paper covers rock

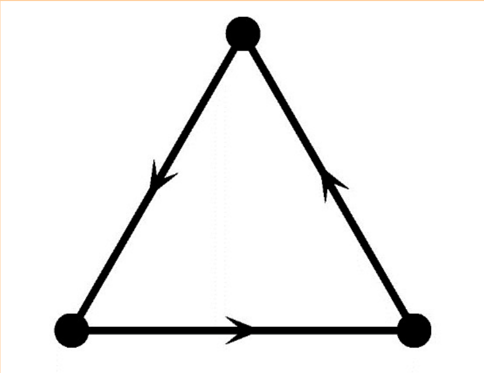
cyclic dominance

$$s_x\,,\,s_y = \begin{pmatrix}1\\0\\0\end{pmatrix},\quad \begin{pmatrix}0\\1\\0\end{pmatrix},\quad \begin{pmatrix}0\\0\\1\end{pmatrix}$$

Payoff matrix (zero-sum game):

$$\mathbf{A} = \begin{pmatrix}0 & 1 & -1\\ -1 & 0 & 1\\ 1 & -1 & 0\end{pmatrix}$$

Adjacency matrix of this directed graph:



Nash equilibrium: mixed strategy

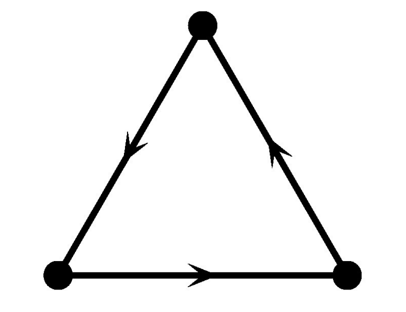
$$s_x^*\,,\,s_y^* = \begin{pmatrix}1/3\\1/3\\1/3\end{pmatrix}$$

Evolutionary rock–paper–scissors model on the square lattice Tainaka 1988

cyclic predator–prey model with simplified evolutionary rule

Individuals are located at the sites x of a square lattice (periodic boundary conditions)

s_{x}=1, 2, 3 (strategy = species 1, 2, and 3)



Random distribution of strategies in the initial state.

Evolutionary rule (invasion between nearest neighbours):
- we choose a site x and its neighbour y at random
- (s_x,s_y) pair transforms into $(s_{\mathrm{x}},s_{\mathrm{x}})$, if s_{x} is the predator of s_{y}
  or $(s_{\mathrm{y}},s_{\mathrm{y}})$, if s_{y} is the predator of s_{x}
- nothing happens if $s_x=s_y$

Simulation: self-organizing pattern

simulation

- the three strategies alternate cyclically at each site
- the local oscillation cannot be synchronized by short range interactions
- the dynamical balance is sustained by cyclic invasions

Simulations on the square lattice

Strategy frequencies: 1/3

Correlation functions:

equal-time: independent of y and k

$n_k(x,t)$: occupation number

equal-position : t-dependence

Numerical results: $C(x) \approx e^{-x/\xi}$, ξ =2.8(2)

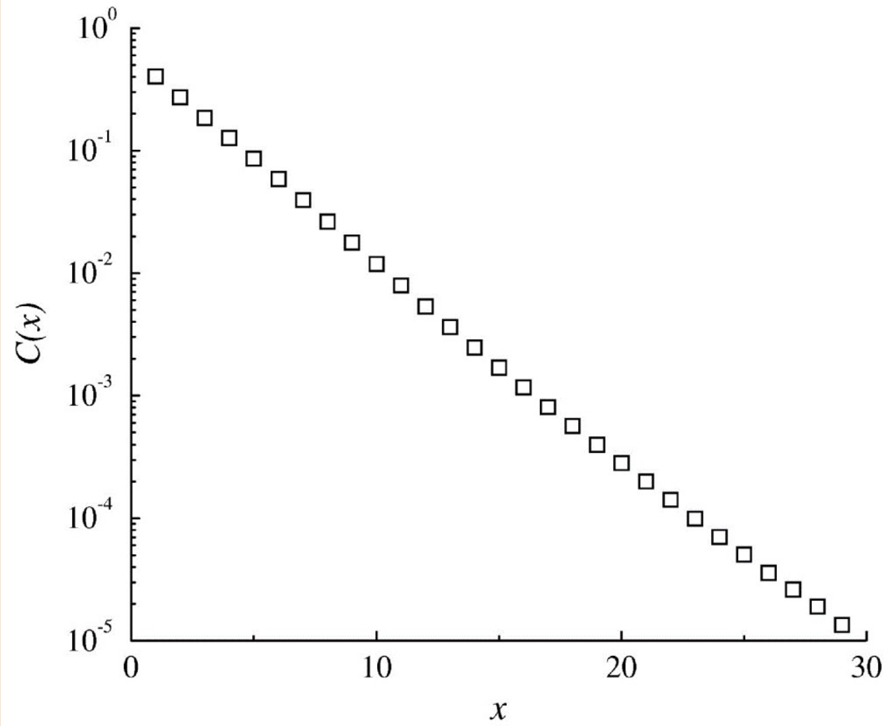
Notation: average value of $A = \langle A \rangle$

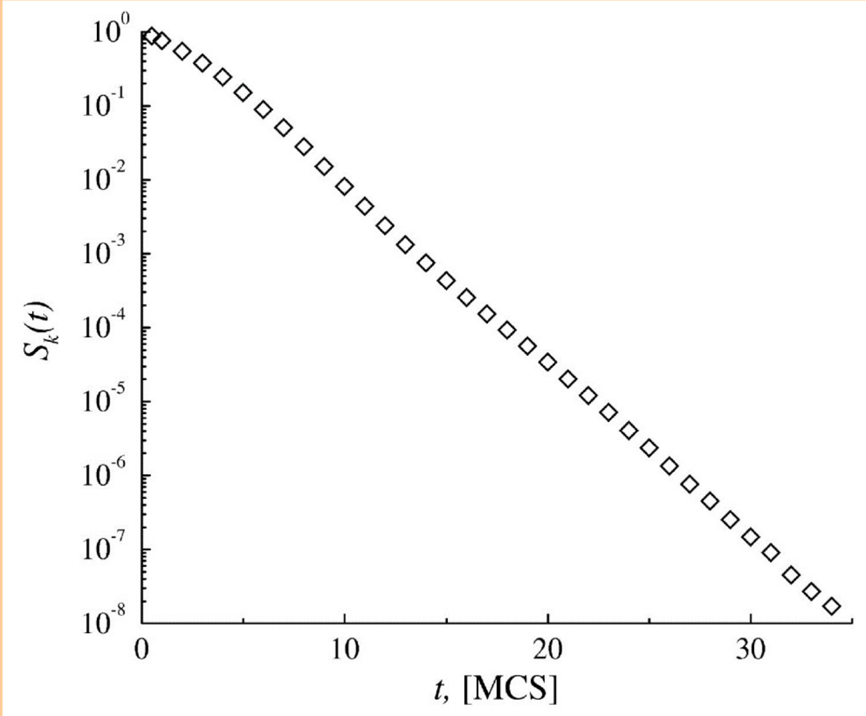
$$C_k(x) = \langle n_k(y,t) n_k(y+x,t) \rangle - \langle n_k(y,t) \rangle \langle n_k(y+x),t \rangle ,$$

where $n_k(x,t) = \begin{cases} 1 & \text{if} \quad s_x(t) = k \quad (k = 1,2,3) \\ 0 & \text{otherwise} \end{cases}$

$$S_k(t) = \langle n_k(x,t_0) n_k(x,t_0+t) \rangle - \langle n_k(x,t_0) \rangle \langle n_k(x,t_0+t) \rangle$$

$S(t) \approx e^{-t/\tau}$, τ=1.8(1); typical exponential decrease





Mean-field approximation or population dynamics

ρ_s is the frequency of strategy s (s=1, 2, 3)

Equations of motion:

$$\dot{\rho}_1 = \rho_1(\rho_2 - \rho_3)$$
$$\dot{\rho}_2 = \rho_2(\rho_3 - \rho_1)$$
$$\dot{\rho}_3 = \rho_3(\rho_1 - \rho_2)$$

Conserved quantities:

$$\dot{\rho}_1 + \dot{\rho}_2 + \dot{\rho}_3 = 0, \quad \Rightarrow \quad \rho_1 + \rho_2 + \rho_3 = 1$$

and

$$\frac{\dot{\rho}_1}{\rho_1} + \frac{\dot{\rho}_2}{\rho_2} + \frac{\dot{\rho}_3}{\rho_3} = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\ln \rho_1 + \frac{\mathrm{d}}{\mathrm{d}t}\ln \rho_2 + \frac{\mathrm{d}}{\mathrm{d}t}\ln \rho_3 = 0$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\ln \rho_1\rho_2\rho_3 = 0$$
$$\Rightarrow \quad \rho_1\rho_2\rho_3 = C = \text{constant}$$

Stationary solutions of the equations of motion:

symmetric:

small perturbation initiates oscillation

$$\rho_1 = \rho_2 = \rho_3 = \frac{1}{3}$$

3 homogeneous:

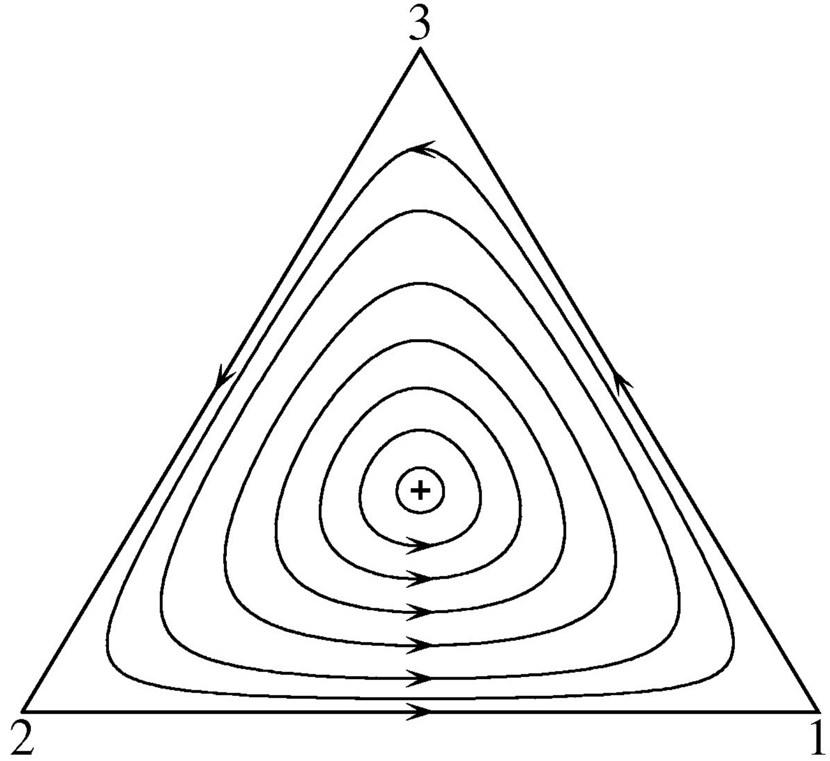
unstable against the corresponding predator

$$\rho_1 = 1,\ \rho_2 = 0,\ \rho_3 = 0$$
$$\rho_1 = 0,\ \rho_2 = 1,\ \rho_3 = 0$$
$$\rho_1 = 0,\ \rho_2 = 0,\ \rho_3 = 1$$

Numerical solutions:

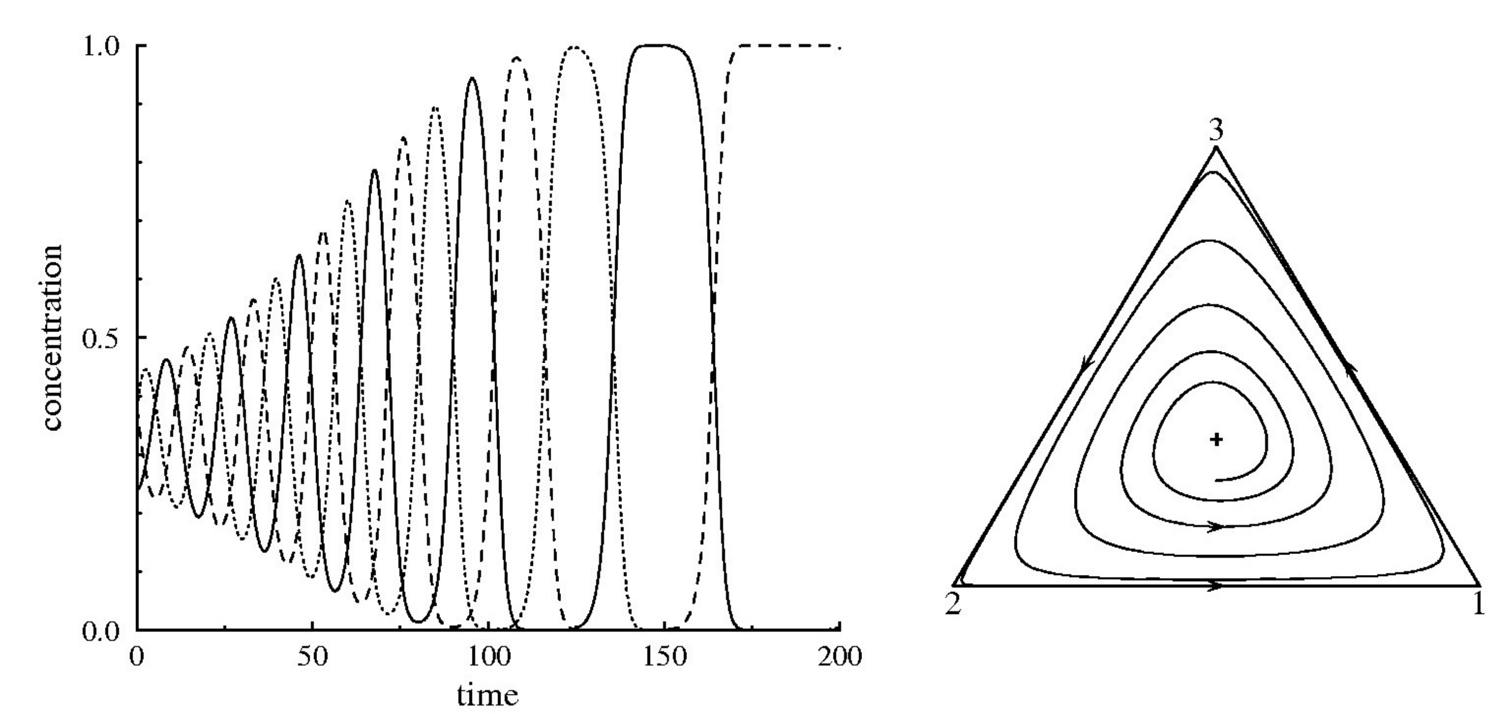
oscillation in ρ_s

concentric trajectories on the simplex



Pair approximation (numerical integration of the equations of motion)

Oscillations in the strategy frequencies/concentrations
with growing amplitudes and periodic times.



The trajectories spiral out and approach the edges of the triangle.
Numerical (rounding) errors stop the evolution in one of the homogeneous solutions.

This approximation cannot reproduce the results of MC simulation on the square lattice.
However, the more sophisticated 4-site approximation works well.

Cyclic predator–prey model (rock–paper–scissors game) on the Bethe lattice (z=3)

Monte Carlo simulation (on random regular graph for large N):

oscillation $\rightarrow$ limit cycle (thick line)

Mean-field appr. predicts: periodic trajectories

Pair appr. predicts: growing oscillation (ending in one of the homogeneous states)

6-site appr. predicts: tending toward a limit cycle (dashed line)

Monte Carlo simulations:

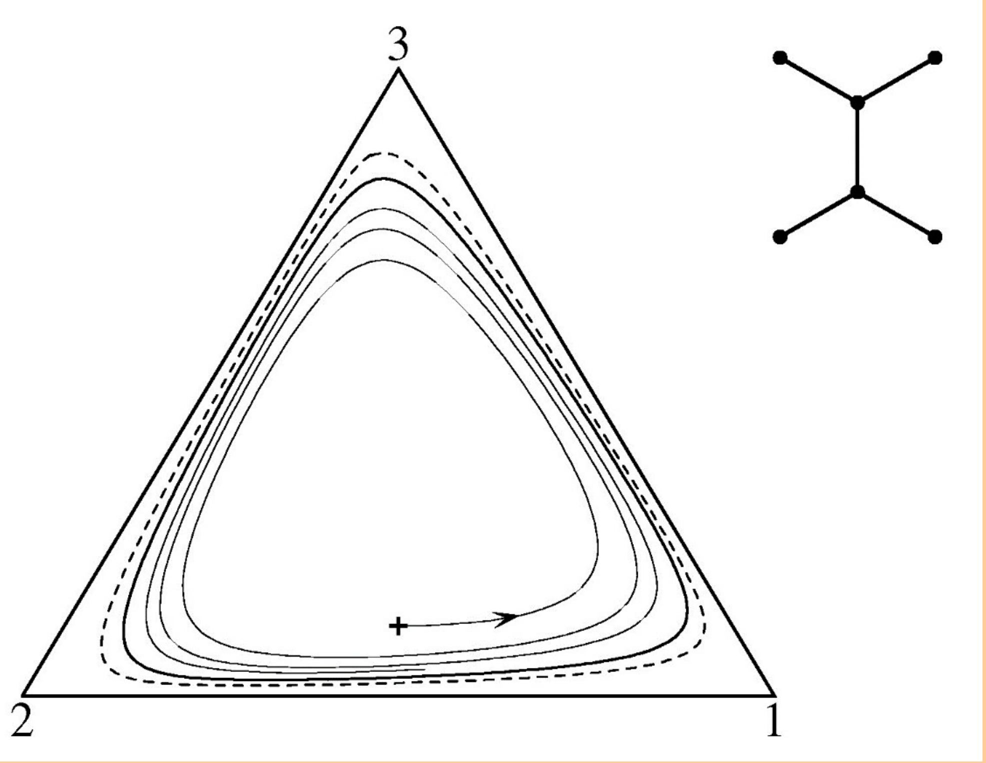
Similar behaviour for z=4.

Prediction of pair appr. is observed for $z \geq 6$.

Global oscillation can also be observed for small-world connections if the players choose (arbitrarily distant) coplayers with a probability of Q.

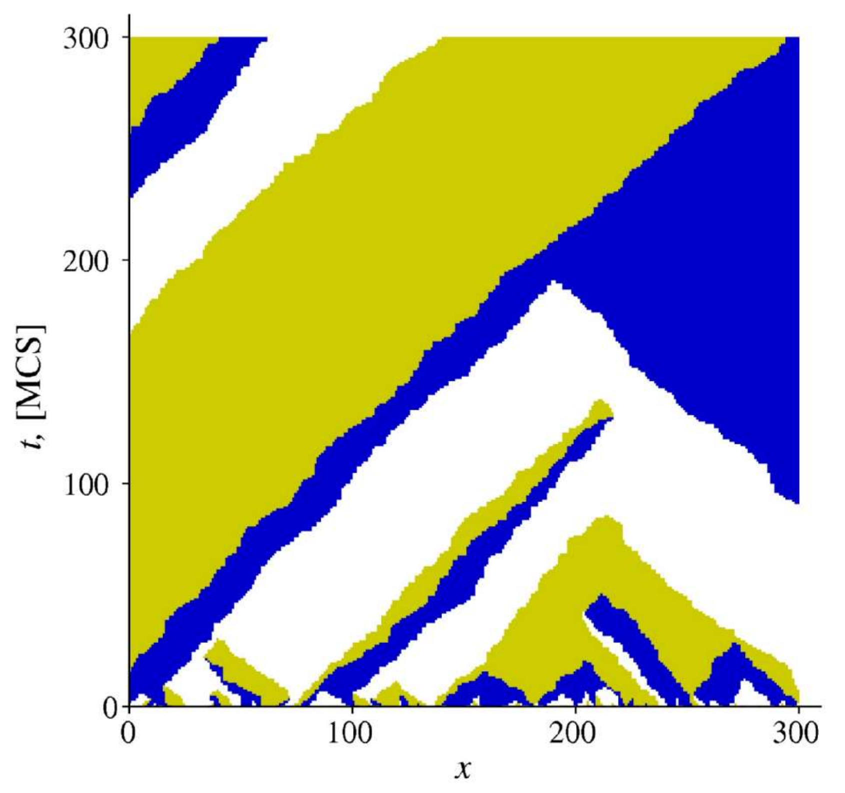
This is also a „small-world effect”.

Q=0.08



Three-strategy cyclic model on the one-dimensional lattice (z=2)

Simulation: evolution from a random initial state $\rightarrow$ ordering process(es)



Two types of domains are distinguished:

1.) Homogeneous domains with size:

$$l \approx t^{3/4}$$

derived from scaling laws

2.) Superdomains (set of hom. domains)

with left (or right) moving fronts

with size: $h \approx t$

Pair appr. is exactly solvable for z=2 if we assume several symmetries:

$$p_1(1) = p_1(2) = p_1(3) = 1/3,$$

$$p_2(1,2) = p_2(2,3) = p_2(3,1) = \rho_r$$

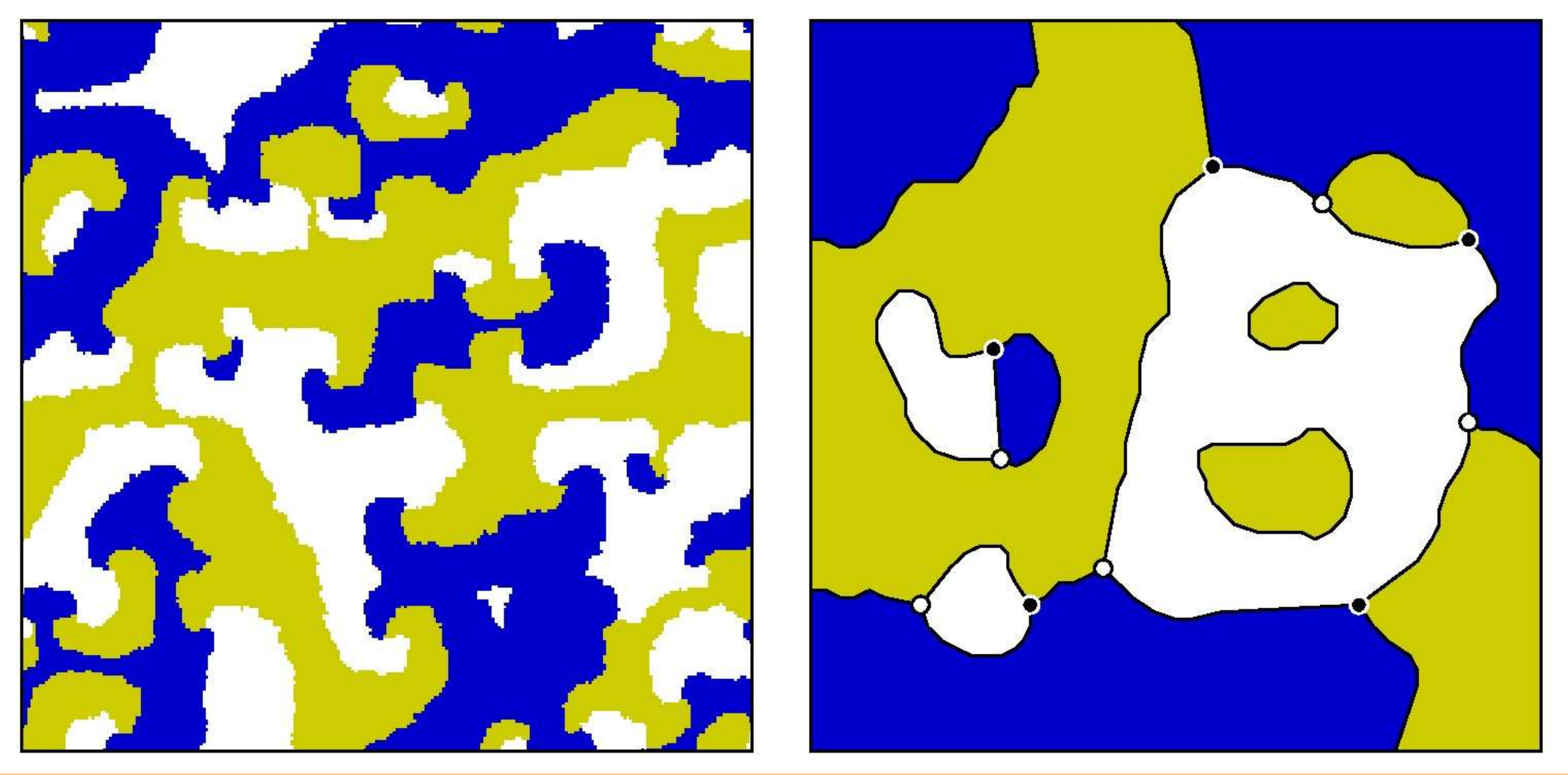
$$p_2(2,1) = p_2(3,2) = p_2(1,3) = \rho_l$$

$$\Rightarrow \rho_r(t) = \rho_l(t) \approx \frac{1}{3+3t}$$

Rotating spiral arms (vortices) in two-dimensional systems

General topological features:

simulation



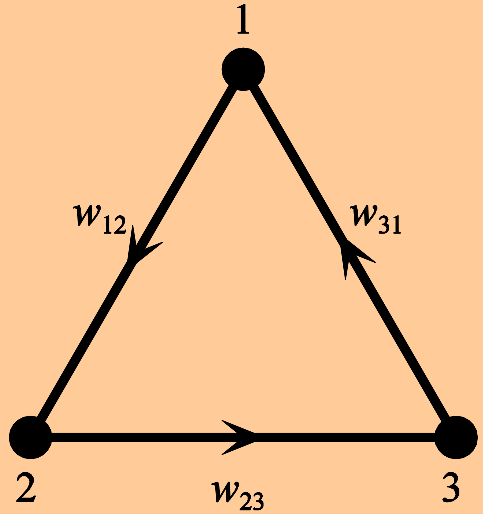
Rotating vortex–antivortex pairs are created that block domain growth.

Three-edge vertices and antivertices are located alternately along the domain boundaries.

Model with different invasion rates

Equations of motion (mean-field appr.):

$$\dot{\rho}_1 = w_{12}\rho_1\rho_2 - w_{31}\rho_3\rho_1$$
$$\dot{\rho}_2 = w_{23}\rho_2\rho_3 - w_{12}\rho_1\rho_2$$
$$\dot{\rho}_3 = w_{31}\rho_3\rho_1 - w_{23}\rho_2\rho_3$$



Stationary solutions:

- coexistence: $$\rho_1 = \frac{w_{23}}{w_{12}+w_{23}+w_{31}},\ \rho_2 = \frac{w_{31}}{w_{12}+w_{23}+w_{31}},\ \rho_3 = \frac{w_{12}}{w_{12}+w_{23}+w_{31}}$$

- Homogeneous states: (unstable)

$$\rho_1 = 1,\ \rho_2 = 0,\ \rho_3 = 0$$
$$\rho_1 = 0,\ \rho_2 = 1,\ \rho_3 = 0$$
$$\rho_1 = 0,\ \rho_2 = 0,\ \rho_3 = 1$$

Conserved quantities:

$$\rho_1 + \rho_2 + \rho_3 = 1$$
$$\rho_1^{w_{23}}\rho_2^{w_{31}}\rho_3^{w_{12}} = C = \text{constant}$$

Conclusions

- All three strategies survive (stabilizing effect).
- **Notice the unexpected phenomenon**:

  The increase of one of the invasion rates will be beneficial to the predator

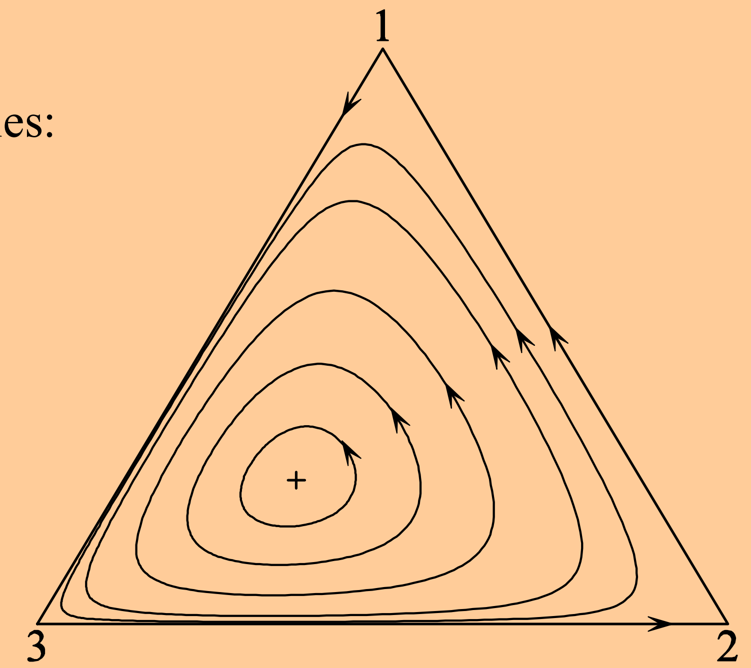
  of the favoured (by w_{ij}) species.

  A similar phenomenon occurs for other types of external effects.

  This robust behaviour can also be observed in spatial models (see page 2).

  Explanation: The food web mediates this effect along directed loops with odd edges.

- Asymmetric oscillation
  with conserved quantities:



Potential game + weak cyclic dominance under logit dynamics

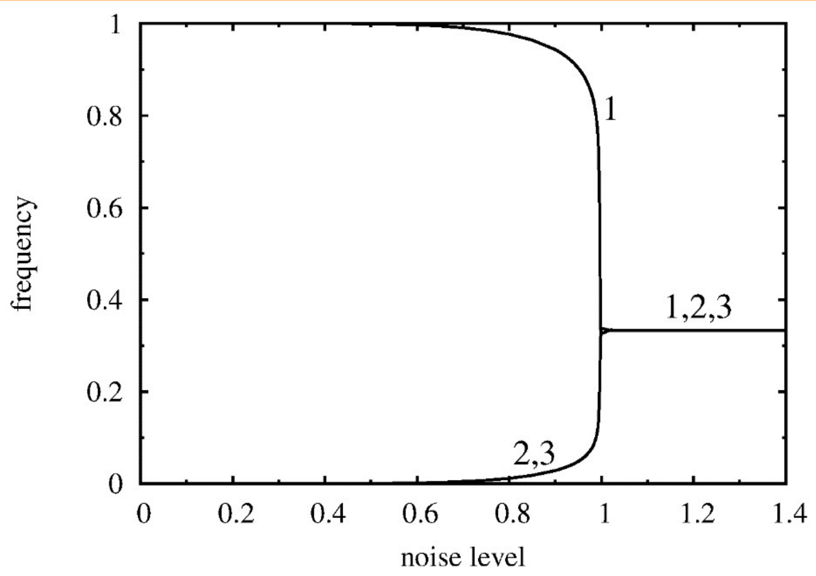
Consider a linear combination of three elementary games:

$$\mathbf{A} = \begin{pmatrix} 1+\varepsilon & \varepsilon-\lambda & \varepsilon+\lambda \\ \lambda & 1 & -\lambda \\ -\lambda & \lambda & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \varepsilon \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \lambda \begin{pmatrix} 0 & -1 & 1 \\ 1 & 0 & -1 \\ -1 & 1 & 0 \end{pmatrix}$$

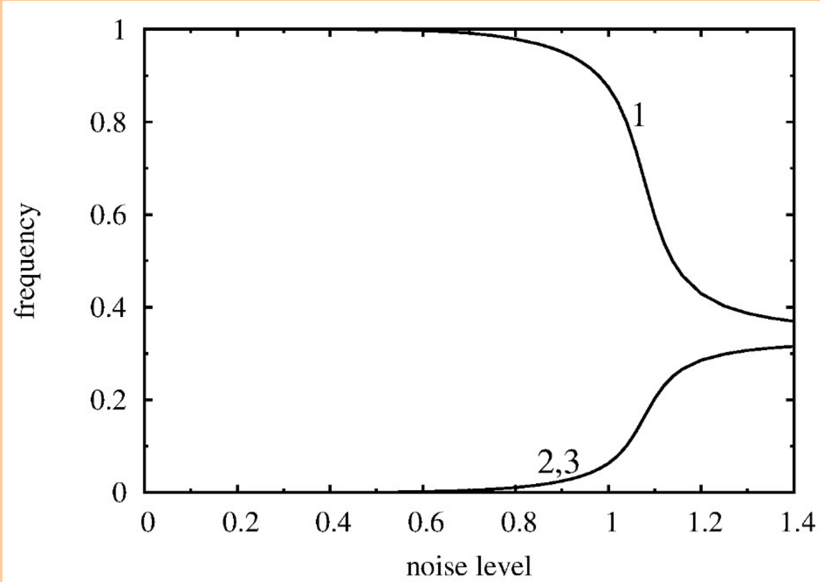
- three-state Potts model (ε=λ=0)
- external field supporting strategy 1 (R) if ε>0 and λ=0
- rock–paper–scissors game representing cyclic dominance 1→3→2→1

Potts model (λ=0): $\rho_i(K)$

when ε=0



when ε=0.01



Potts model + RPS game (ε=0) (continued)

random initial state → growing domains with point defects → rotating spiral arms

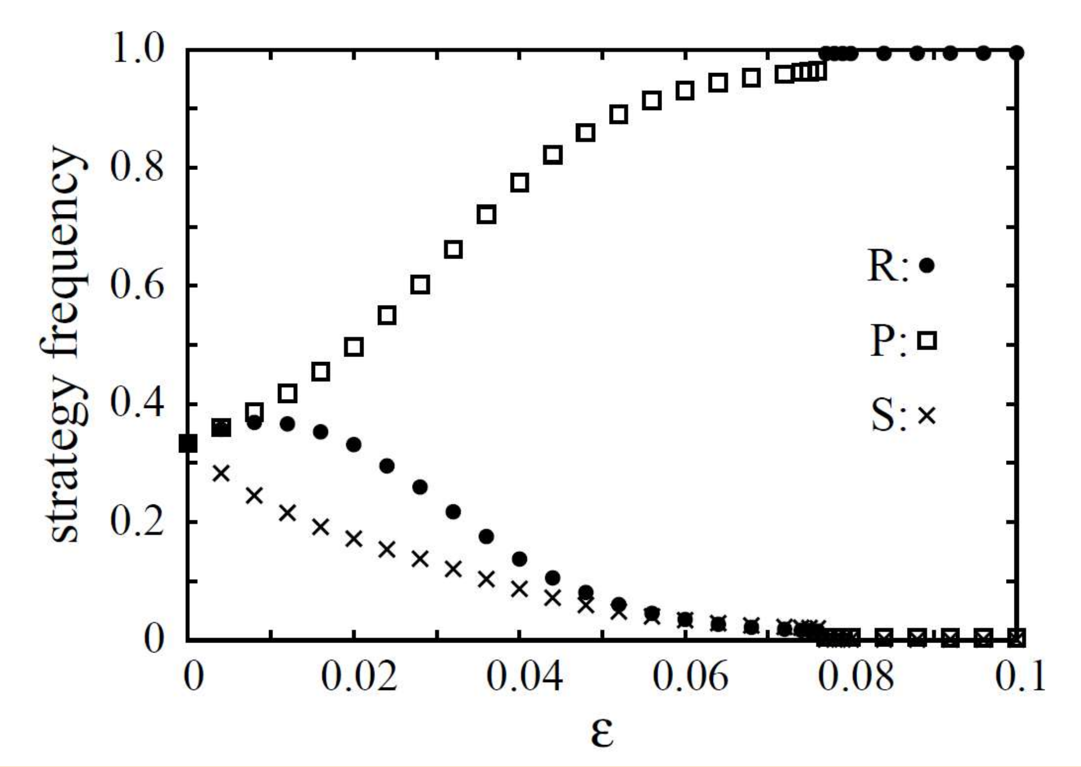
In the stationary state: ρ_i=1/3.

Average domain size is proportional to ~1/λ.

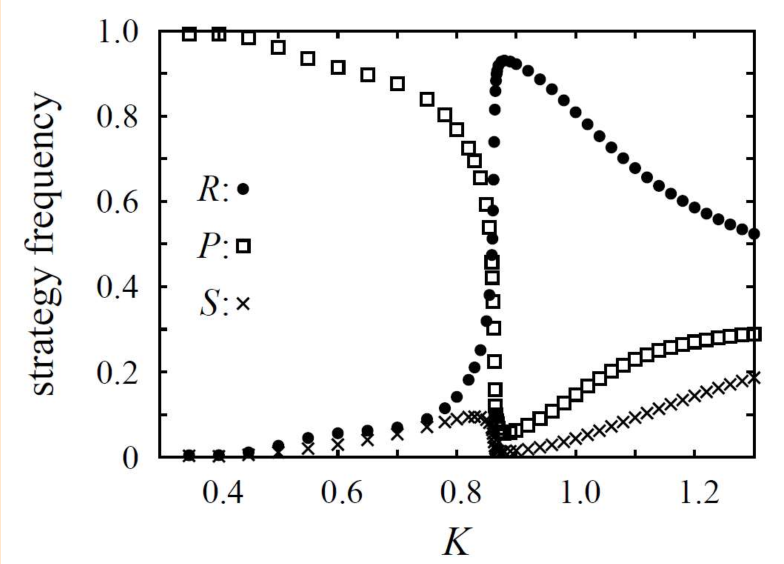
$\rho_i(\varepsilon)$ for λ=0.1 and K=0.7

For weak ε:
The external support ultimately benefits *P.*
(Tainaka effect)

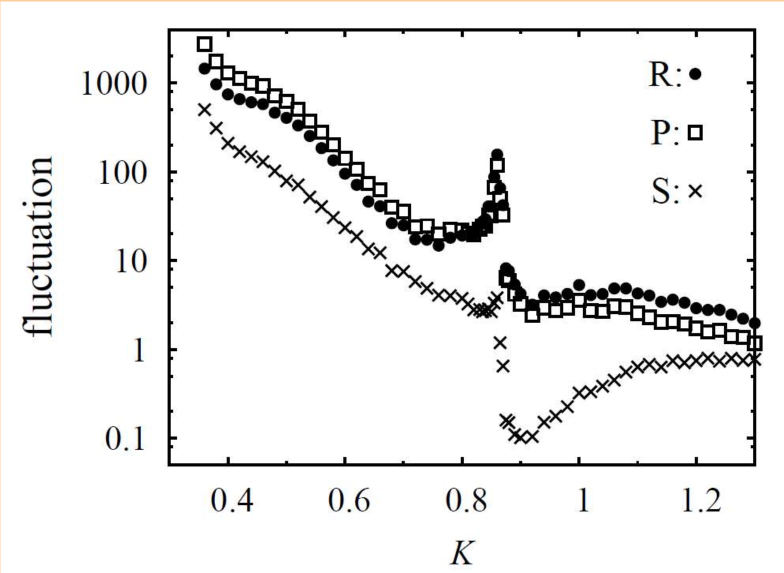
For strong ε:
Brute force eventually results in the dominance of *R.*



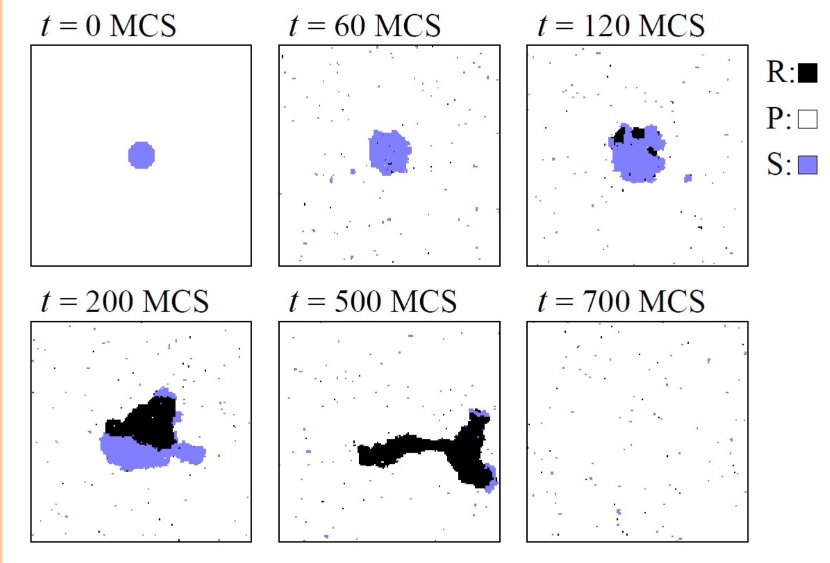
Noise-dependence of strategy frequencies
when ε=0.05 and λ=0.1

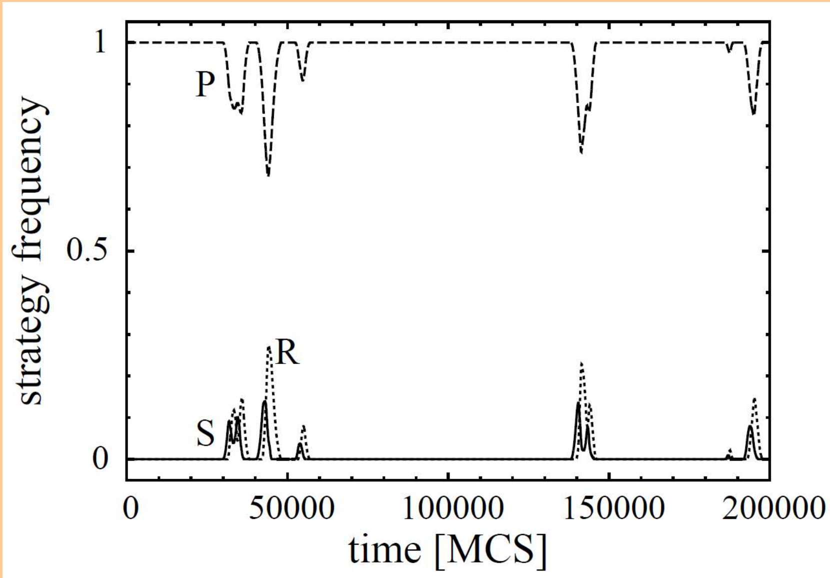


Huge fluctuations in strategy frequencies



Avalanches at low noises
Increasing and shrinking domains burst
when ε=0.05, λ=0.1, and K=0.6





Home assignment

12.1. Determine the periodic time of oscillations in the population dynamics of the rock–paper–scissors game in the $\varepsilon \to 0$ limit, if $\rho_1(t{=}0)=1/3+2\varepsilon$ and $\rho_2(t{=}0)=\rho_3(t{=}0)=1/3-\varepsilon$! (Advice: Use the formalism of complex functions, that is, $\delta\rho_k(\mathrm{t})=\rho_k(\mathrm{t})-1/3=\varepsilon \mathrm{e}^{\mathrm{i}\omega t+\mathrm{i}2\pi(k-1)/3}$ +c.c.).