Methods for studying spatial evolutionary games **Lecture 11**

1) Monte Carlo simulations

2) Methods of equilibrium statistical physics

3) Methods of non-equilibrium statistical physics

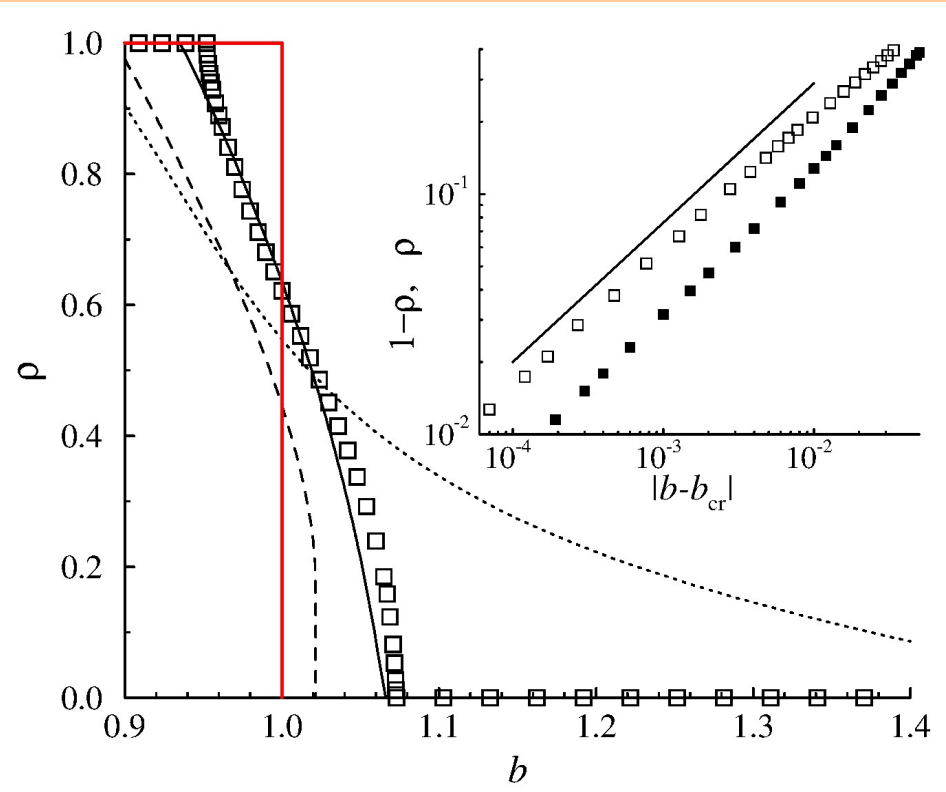
Comparison of results (PD game, square lattice, pairwise imitation, K=0.4)

Red: MF; dotted: pair (or 2-site); dashed: 4-site; solid: 9-site approximation

Squares: MC results

Approximative analytical treatments help us

- extract general statements
- identify universal features
- quickly derive initial results



Evaluation of cluster configuration probabilities

extension of mean-field and pair approximations

N players are located at the sites (x) of a lattice with PBCs.
Symmetries: translation, rotation, and reflection
Each player follows one of Q strategies:

$$s_x = 1, \cdots, Q.$$

The probability of finding strategy s_1 is identical at all the sites and it is denoted by

$$p_1(s_1).$$

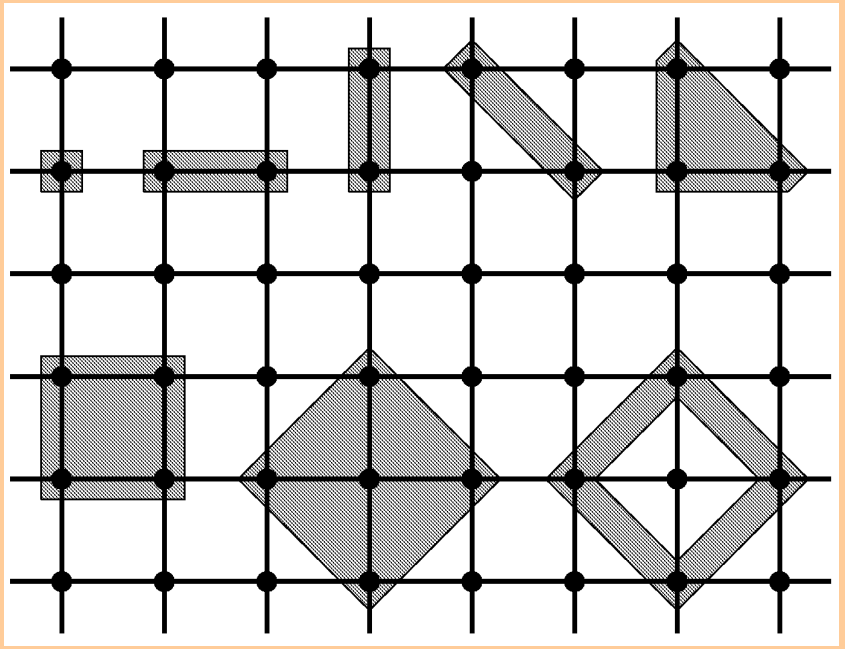
(Time-dependence is not indicated.)
Configuration probabilities on two sites:

$$p_2(s_1, s_2)$$

Configuration probabilities on n-site clusters:

$$p_n(s_1, s_2, \ldots, s_n)$$

Number of configurations: Q^n.



Compatibity conditions

Normalization:

$$\sum_{s_1=1}^{Q} p_1(s_1) = 1$$

The configuration probabilities are not independent of each other.

For example, if n=2, then

$$\sum_{s_2=1}^{Q} p_2(s_1, s_2) = \sum_{s_2=1}^{Q} p_2(s_2, s_1) = p_1(s_1)$$

For arbitrary n:

$$\sum_{s_{n+1}=1}^{Q} p_{n+1}(s_1, \ldots, s_n, s_{n+1}) = p_n(s_1, \ldots, s_n)$$

Dependence on the form of the cluster is not indicated in this notation.

Usually, compact clusters are used.

Consequences of compatibility conditions

Number of free parameters is reduced.
For example, for one- and two-site clusters and Q=2:

$$p_1(1) = c$$
$$p_1(0) = 1 - c$$
$$p_2(1,1) = q$$
$$p_2(1,0) = p_2(0,1) = c - q$$
$$p_2(0,0) = 1 - 2c + q$$

because

$$p_2(1,0) + p_2(1,1) = p_2(0,1) + p_2(1,1) = p_1(1),$$
$$p_2(1,0) + p_2(0,0) = p_2(0,1) + p_2(0,0) = p_1(0).$$

$p_2(1,0) = p_2(0,1)$ follows from compatibility

Consequence:

The 6 configuration probabilities are defined by just 2 parameters.

Larger clusters can exhibit further symmetries.

Configuration probabilities can be built up using Bayesian approximation

Namely, the conditional probability of finding s_2 at site x_2 given that x_1 follows strategy s_1 is

$$p_1(s_2 \mid s_1) = \frac{p_2(s_1, s_2)}{p_1(s_1)}$$

So for a three-site cluster using the conditional probability of finding s_3 given s_2:

$$p_3(s_1, s_2, s_3) = p_2(s_1, s_2) p_1(s_3 \mid s_1, s_2) \cong p_2(s_1, s_2) \frac{p_2(s_2, s_3)}{p_1(s_2)}$$

And by the same token for a (linear) four-site cluster:

$$p_4(s_1, s_2, s_3, s_4) \cong p_3(s_1, s_2, s_3) \frac{p_2(s_3, s_4)}{p_1(s_3)} \cong p_2(s_1, s_2) \frac{p_2(s_2, s_3)}{p_1(s_2)} \frac{p_2(s_3, s_4)}{p_1(s_3)}$$

Generalization to linear *n*-site clusters:

$$p_n(\{s_x\}) \cong p_2(s_1, s_2) \prod_{i=2}^{n-1} \frac{p_2(s_i, s_{i+1})}{p_1(s_i)}$$

This approximation preserves the compatibily conditions, if d=1,

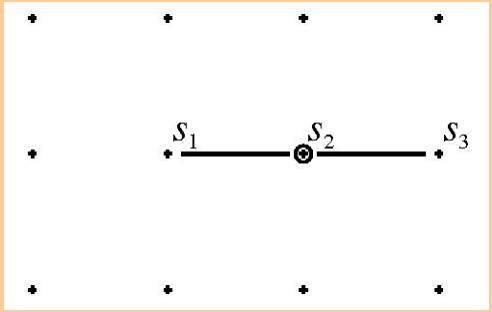
Graphical representation of the products

Configuration probabilities can be represented by geometrical symbols

multiplication (division) by $p_1(s_1)$: solid (empty) circle at site x_1

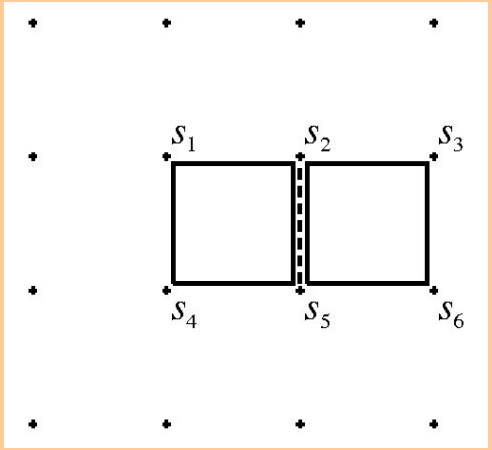
multiplication (division) by $p_2(s_1,s_2)$: solid (dashed) line connecting x_1 and x_2

$$p_3(s_1,s_2,s_3)=\frac{p_2(s_1,s_2)p_2(s_2,s_3)}{p_1(s_2)}$$



Generalization for larger clusters:

$$p_6(s_1,s_2,s_3,s_4,s_5,s_6)=\frac{p_4(s_1,s_2,s_4,s_5)p_4(s_2,s_3,s_5,s_6)}{p_2(s_2,s_5)}$$



These graphs are useful when performing calculations.

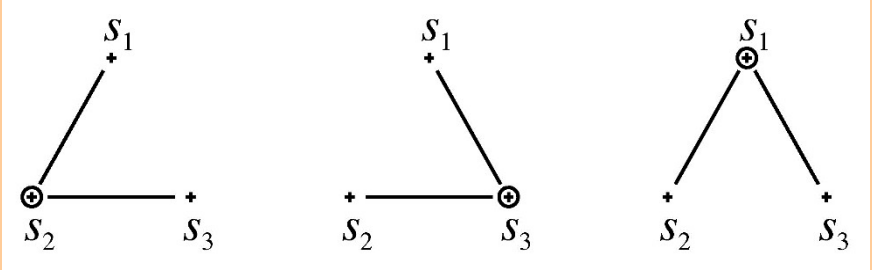
Problem of enclosure

For triangular clusters we have different options:

For these three options:

rotational symmetry is broken,

compatibility conditions are broken, too

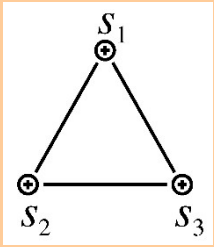


Suggested solutions:

consider all with weight factor of 1/3,

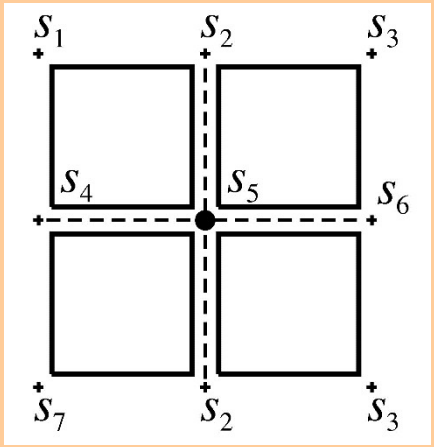
or Kirkwood approximation: $p_3(s_1,s_2,s_3)=\dfrac{p_2(s_1,s_2)p_2(s_2,s_3)p_2(s_3,s_1)}{p_1(s_1)p_1(s_2)p_1(s_3)}$

or use three-site approximations



Similar difficulties can occur for all spatial systems.

an example with a possible solution



These difficulties do not exist on 1D lattices

and loop-free structures (e.g., Bethe lattices)

Approximation of quantities using pair-configuration probabilities:

- Frequency of strategy s in spatial evolutionary games:

$$\rho_s = p_1(s) = \sum_{s_2=1}^{Q} p_2(s, s_2)$$

- Probability of finding homogeneous linear domain of s with a length of n:

$s_i = s$, $i = 1, \ldots, n$

$$p_n(s, \cdots, s) \cong p_1(s) \prod_{i=1}^{n-1} \frac{p_2(s,s)}{p_1(s)} = A \mathrm{e}^{-\frac{n}{\xi}}$$

ξ: correlation length or typical domain size

where $$-\frac{1}{\xi} = \ln\left(\frac{p_2(s,s)}{p_1(s)}\right)$$

- Average payoff for z neighbours:

$$\langle U_x \rangle = z \sum_{s_1, s_2=1}^{Q} p_2(s_1, s_2) A_{s_1, s_2}$$

- Other quantities (e.g., total payoff or entropy) can also be evaluated.

- The method can be extended to larger clusters, too.

Cluster variation method at the level of pair-approximation:

The thermodynamic potential (average potential + $K\times$entropy) can be given as a function of c and q (when Q=2) in the $N\to\infty$ limit as:

$$A(c,q) = N \cdot a(c,q)$$

According to the extremum principles the equilibrium state can be determined by solving the following equations:

$$\frac{\partial a(c,q)}{\partial c} = 0 \text{ and } \frac{\partial a(c,q)}{\partial q} = 0.$$

For example, on the 1D lattice:

$$\langle \Phi \rangle = U + KS =$$

$$\sum_{s_1,s_2=1}^{Q} p_2(s_1,s_2) V_{s_1,s_2} + K\left\{ -\sum_{s_1,s_2=1}^{Q} p_2(s_1,s_2)\ln[p_2(s_1,s_2)] + \sum_{s_1=1}^{Q} p_1(s_1)\ln[p_1(s_1)] \right\}$$

where U and S denote the specific potential and entropy.

The method can be extended to larger clusters, too.

Dynamical cluster method

Example: spreading of infection on a 1D lattice (Contact Process)

$$s_x = \begin{cases} 0 & \text{healthy} \\ 1 & \text{infected} \end{cases} \qquad x = 1, \ldots, N, \quad N \to \infty. \text{ (PBC)}$$

The dynamics is controlled by repeating the following elementary processes at site x:

recovery: $s_x=1 \to 0$ with a probability of $1/(1+\lambda)$

infection: $s_x=0 \to 1$ with a probability of $\lambda/(1+\lambda)$

if its randomly selected neighbour is infected

Parametrization of configuration probabilities:

$$p_1(1) = c$$
$$p_1(0) = 1 - c$$
$$p_2(1,1) = q$$
$$p_2(1,0) = p_2(0,1) = c - q$$
$$p_2(0,0) = 1 - 2c + q$$

Contributions of the elementary processes to the time derivatives of the configuration probabilities:

z=2 and the denominator is eliminated by choosing a suitable time scale:

$1 \to 0$: $\dot{p}_1(1) = -p_1(1)$ and $\dot{p}_1(0) = p_1(1)$

$101 \to 111$: $\dot{p}_1(1) = \lambda p_3(1,0,1)$ ...

$100 \to 110$: $\dot{p}_1(1) = \frac{\lambda}{2} p_3(1,0,0)$ ...

$001 \to 011$: $\dot{p}_1(1) = \frac{\lambda}{2} p_3(0,0,1)$ ...

Total contribution:

$$\dot{p}_1(1) = -p_1(1) + \frac{\lambda}{2}\left[2p_3(1,0,1) + p_3(0,0,1) + p_3(1,0,0)\right]$$

$$\dot{p}_1(1) = -p_1(1) + \frac{\lambda}{2}\left[p_2(1,0) + p_2(0,1)\right]$$

$$\dot{p}_1(1) = -p_1(1) + \lambda p_2(1,0), \quad \text{because} \quad p_2(1,0) = p_2(0,1) \quad \text{(compatibility)}$$

and $\dot{p}_1(0) = -\dot{p}_1(1)$

Contributions to the two-site configuration probabilities:

$11 \to 01$: $\dot{p}_2(1,1) = -p_2(1,1)$ and $\dot{p}_2(0,1) = p_2(1,1)$

$11 \to 10$: $\dot{p}_2(1,1) = -p_2(1,1)$ and $\dot{p}_2(1,0) = p_2(1,1)$

$101 \to 111$: $\dot{p}_2(1,1) = 2\lambda p_3(1,0,1)$ ...

$100 \to 110$: $\dot{p}_2(1,1) = \frac{\lambda}{2} p_3(1,0,0)$ ...

$001 \to 011$: $\dot{p}_2(1,1) = \frac{\lambda}{2} p_3(0,0,1)$ ...

The sum of the above contributions

$$\dot{p}_2(1,1) = -2p_2(1,1) + \lambda\left[p_2(1,0) + p_3(1,0,1)\right]$$

As we have only two parameters (c and q) we do not need to derive further equations of motion.

Notice: The above set of EoM is not closed/solvable due to the additional terms (e.g., $p_3(1,0,1)$].

Consequently, we need further equations

or we need to use the Bayesian approximation formula for quantities like $p_3(s_1,s_2,s_3)$.

One-site (mean-field) approximation

The equations are simplified by $p_n(s_1,\ldots,s_n) = p_1(s_1)p_1(s_2)\cdots p_1(s_n)$

Then:

$$\begin{aligned}\dot{p}_1(1) &= -p_1(1) + \lambda p_2(1,0) \\ &= -p_1(1) + \lambda p_1(1)p_1(0) \\ \dot{c} &= -c + \lambda c(1-c)\end{aligned}$$

Stationary solution: $\dot{c} = 0$

$$0 = -c + \lambda c(1-c)$$

Trivial solution: $c = 0$

Non-trivial solution:

$$\begin{aligned}0 &= -1 + \lambda(1-c) \\ \lambda c &= \lambda - 1 \\ c &= \frac{\lambda - 1}{\lambda}\end{aligned}$$

Two-site (pair) approximation

$$p_n(s_1,\ldots,s_n) = p_2(s_1,s_2)\prod_{x=2}^{n-1}\frac{p_2(s_x,s_{x+1})}{p_1(s_x)}$$

The following two equations are to be solved:

$$\dot{p}_1(1) = -p_1(1) + \lambda p_2(1,0)$$
$$\dot{p}_2(1,1) = -2p_2(1,1) + \lambda p_2(1,0) + \lambda p_3(1,0,1)$$

Plugging in the parametrization and the approximation, we get

$$\dot{c} = -c + \lambda(c-q)$$

$$\dot{q} = -2q + \lambda(c-q) + \lambda\frac{(c-q)^2}{1-c}$$

Stationary solution: $\dot{c} = 0$ and $\dot{q} = 0$

$$c = \lambda(c-q) \qquad \Rightarrow \qquad q = \frac{\lambda-1}{\lambda}c$$

$$2q = \lambda(c-q) + \lambda\frac{(c-q)^2}{1-c}$$

$$\frac{2(\lambda-1)}{\lambda}c = c + \frac{1}{\lambda}\frac{c^2}{(1-c)}$$

Trivial solution: $c = 0$ and $q = 0$

Non-trivial solution:

$$\frac{2(\lambda-1)}{\lambda} = 1 + \frac{1}{\lambda}\frac{c}{(1-c)}$$

$$2(\lambda-1)(1-c) = \lambda(1-c) + c$$

$$2\lambda - 2 - 2(\lambda-1)c = \lambda - (\lambda-1)c$$

$$\lambda - 2 = (\lambda-1)c$$

$$\Rightarrow \quad c = \frac{\lambda-2}{\lambda-1}$$

and $$q = \frac{\lambda-1}{\lambda}c = \frac{\lambda-1}{\lambda}\frac{\lambda-2}{\lambda-1} = \frac{\lambda-2}{\lambda}$$

Stability analysis: The active (c>0) solution is stable if λ>2.

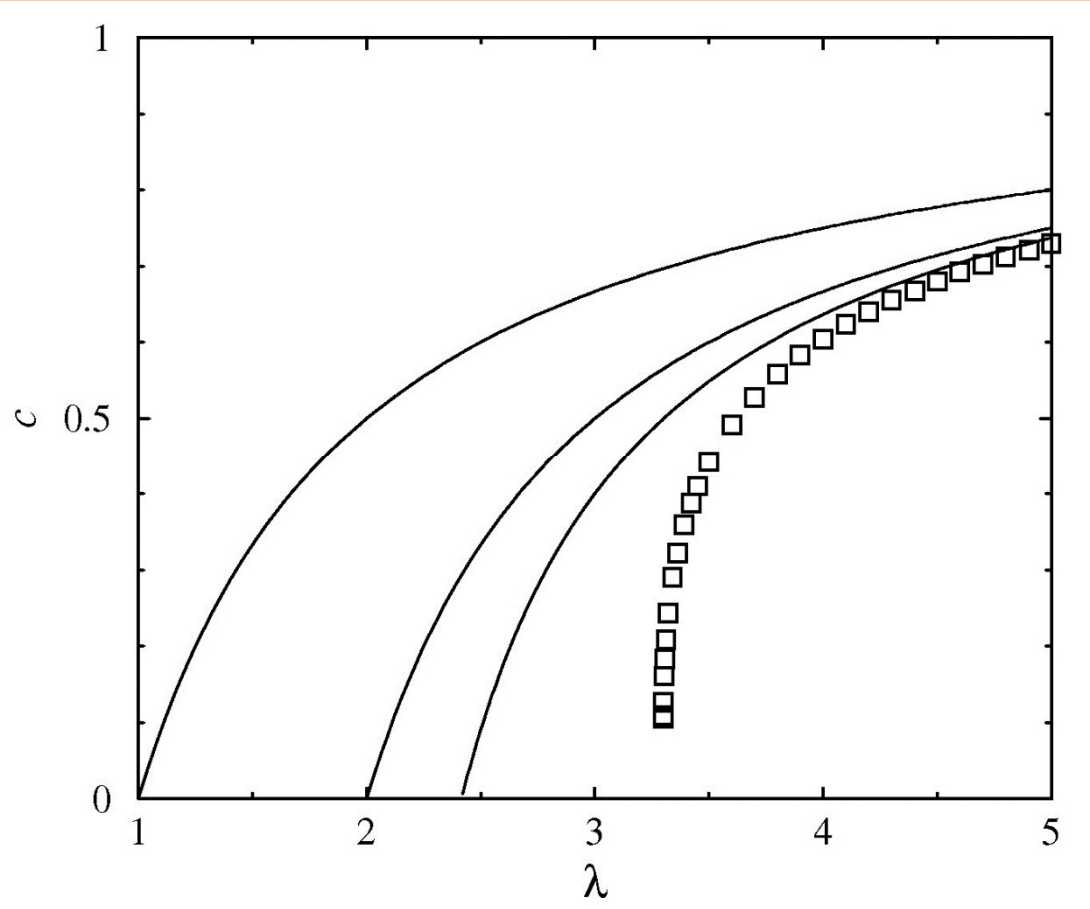
The absorbing (trivial, c=0) solution is stable if λ<2

The calculations can be performed when using larger clusters:

We can develop algorithms for the derivation of the EoM.

numerical solution use of ”recipes” is suggested

Comparison of results (1-, 2-, and 3-site approximation and MC simulation) in 1D



Increasing *n* improves the accuracy.

Simulation:

Directed percolation (DP) type critical transition, that is,

correlation length, relaxation time, and fluctuations diverge when $\lambda \rightarrow \lambda_c$.

Numerical solution of the EoM

$\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x})$ Q variables and eqs.

In the stationary state: $\dot{\mathbf{x}} = \mathbf{F}(\mathbf{x}) = 0$

1. Solution by iteration (Newton's method): $\mathbf{x}_0 \rightarrow \mathbf{x}_1 \rightarrow \mathbf{x}_2 \rightarrow$ etc.

$$\mathbf{x}_{n+1} = \mathbf{x}_n - \left(\frac{\partial \mathbf{F}(\mathbf{x})}{\partial \mathbf{x}} \right)^{-1} \Bigg|_{\mathbf{x}=\mathbf{x}_n} \mathbf{F}(\mathbf{x}_n)$$

Use software packages.
Use variables with the highest accuracy.
The iteration converges to the real solution if it is started from a suitable x_0.

2. Numerical integration:

Repeat the following steps:

$$\mathbf{x}(t + dt) = \mathbf{x}(t) + \mathbf{F}(\mathbf{x}(t))dt$$

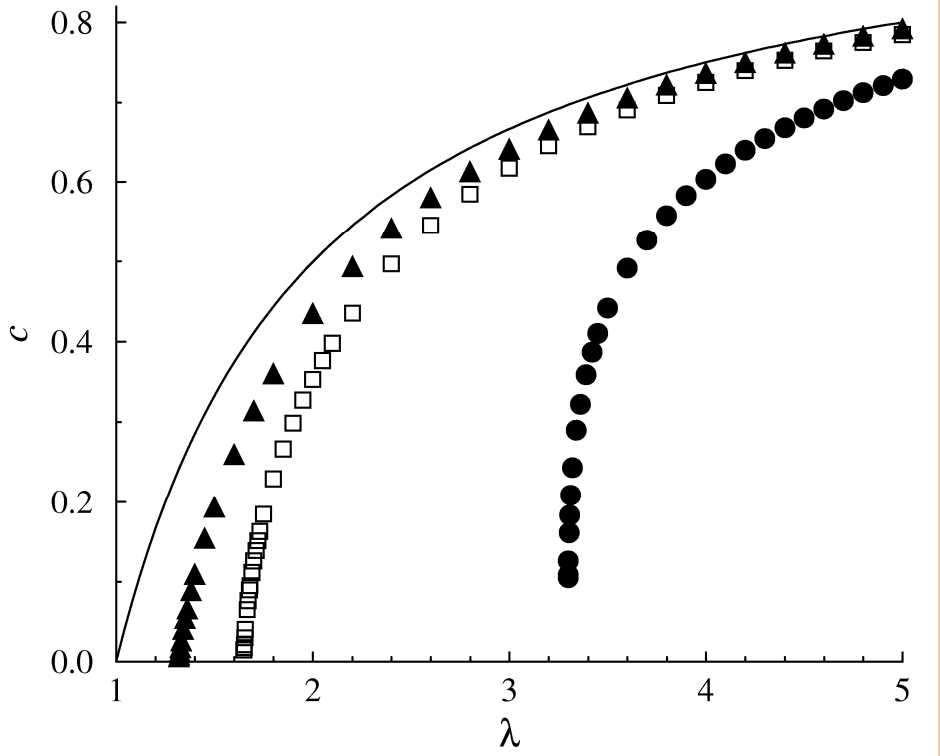
This is the only possibility for limit cycles.
dt can be tuned during the iteration.

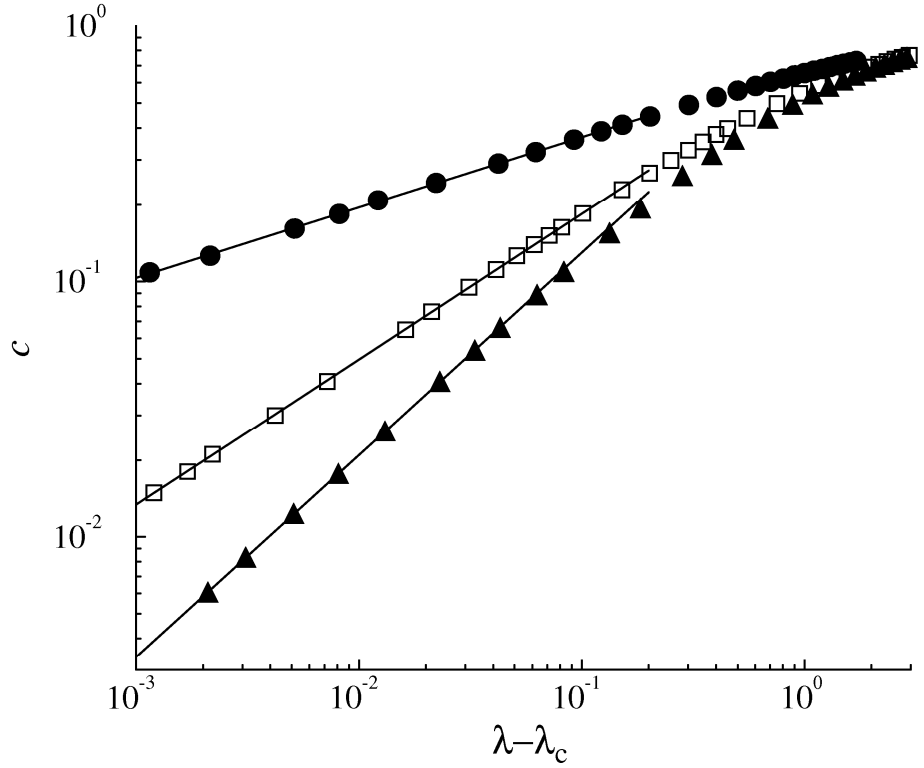
Contact process on *d*-dimensional lattices

Simulation results:

d=1(●), 2(□), and 3(▲)

1-site approximation: solid line, critical behaviour in the vicinity of λ_c





General features of this critical transition from ρ=0 to the active (ρ>0) phase if $N \to \infty$.

Quantities	exponents
Order parameter:	$\rho = \langle s(\mathbf{x},t) \rangle \propto (\lambda - \lambda_c)^{\beta}$
Fluctuation of order parameter	$\chi = L^d \langle [s(\mathbf{x},t) - \langle s(\mathbf{x},t) \rangle]^2 \rangle \propto \lvert \lambda - \lambda_c \rvert^{-\gamma}$
Correlation function:	$G(\mathbf{x},t) = \langle s(\mathbf{x}+\mathbf{x}_0, t+t_0) s(\mathbf{x},t) \rangle - \langle s(\mathbf{x}_0,t_0) \rangle^2$
Equal-time:	$G(\mathbf{x},0) \propto e^{-\lvert x \rvert / \xi}$, $\xi \propto \lvert \lambda - \lambda_c \rvert^{-\nu_m}$
Equal-position:	$G(0,t) \propto e^{-t/\tau}$, $\tau \propto \lvert \lambda - \lambda_c \rvert^{-\nu_p}$

Time-dependent behaviour: spatial spreading of infection from a single site?

Survival of infection:	$S(t) \propto t^{-\delta}$
Average number of survivals:	$n(t) \propto t^{\eta}$
Occupied area (R^2):	$R^2(t) \propto t^{z}$

Exponents for different ds (universal behaviour):

Exp.	d=1	d=2	d=3	d=4
λ_c	3.29785(2)	1.6488(1)	1.3169(1)	
β	0.27649(4)	0.583(4)	0.805(10)	1
γ	0.54386(7)	0.35(1)	0.19(1)	0
ν_p	1.73383(3)	1.295(6)	1.105(5)	1
ν_m	1.09684(6)	0.733(4)	0.581(5)	½
δ	0.15947(3)	0.4505(10)	0.730(4)	1
η	0.31386(3)	0.2295(10)	0.114(4)	0
z	1.26523(3)	1.1325(10)	1.052(3)	1

Mean-field type behavior occurs if $d \geq 4$.

Relations between the exponents can be derived from scaling hypotheses:

$$z = \frac{2\nu_m}{\nu_p}, \quad \delta = \frac{\beta}{\nu_p}, \quad 4\delta + 2\eta = dz, \quad \gamma = d\nu_m - 2\beta, \quad \ldots$$

These general features are characteristic of the DP (directed percolation) universality class.

(these phenomena were first observed there)

DP type critical transition is expected in a system (in the limit $N \to \infty$) if

- there is (at least one) absorbing state
- it involves short range interactions
- MF approximation predicts a similar transition (e.g., for imitation)

(additional symmetries can result in different behaviour)

- the background is homogeneous

inhomogeneous systems can exhibit a Griffiths phase

that leads to slow relaxation

Home assignments

11.1. On a one-dimensional lattice for Q=2, the three-site cluster configuration probabilities satisfy the following relations:

$$p_3(1,0,0) = p_3(0,0,1) \text{ and } p_3(1,1,0) = p_3(0,1,1),$$

whereas for the four-site configuration probabilities

$$p_4(1,1,0,0) \neq p_4(0,0,1,1) \text{ but } p_4(1,0,0,0) = p_4(0,0,0,1).$$

What is the common feature of configurations that can exhibit symmetry breaking for n>4?

11.2. Show that on a one-dimensional lattice for Q=3 the compatibility conditions allow the following relations:

$$p_2(1,0) - p_2(0,1) = p_2(2,1) - p_2(1,2) = p_2(0,2) - p_2(2,0).$$

What is the main feature of the patterns that can be described by the following configuration probabilities:

$$p_2(0,0) = p_2(1,1) = p_2(2,2) = a$$

$$p_2(1,0) = p_2(2,1) = p_2(0,2) = \frac{1}{3} - a$$

$$p_2(0,1) = p_2(1,2) = p_2(2,0) = 0$$

11.3. Find a suitable parametrization for the cluster configuration probabilities on a 2×2 cluster of sites on a square lattice for Q=2, if the system exhibits all the possible symmetries (reflection and rotation)! How many independent parameters does it have?

11.4. Determine the number of parameters we need to introduce at the level of triangular cluster approximation on the kagome and triangular lattices for Q=2!

11.5. Evaluate the specific entropy

$$S = -\frac{1}{N}\sum_{\{s_x\}} p_N(\{s_x\})\ln[p_N(\{s_x\})]$$

in a Q-state one-dimensional system in the $N\to\infty$ limit when

(1) $$p_N(\{s_x\}) = \prod_{x=1}^{N} p_1(s_x),$$

(2) $$p_N(\{s_x\}) = p_1(s_1)\prod_{x=1}^{N-1}\frac{p_2(s_x, s_{x+1})}{p_1(s_x)}.$$