Decomposition of games [see Szabó and Borsos, *Phys. Rep.* 462 (2016) 1] Lecture 7

Symmetric two-person games are defined by a payoff matrix **A**.

For two-strategy games it can be given and decomposed as

$$\mathbf{A} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = a\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + d\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

$$= a\mathbf{e}^{(1)} + b\mathbf{e}^{(2)} + c\mathbf{e}^{(3)} + d\mathbf{e}^{(4)}$$

Each coefficient *a, b, c*, and *d* characterizes the payoff for a specific pure strategy profile and these matrices can be considered as 4-dimensional orthonormal unit vectors by generalizing the traditional concept of the scalar product of two vectors as

$$\mathbf{e}^{(i)} \cdot \mathbf{e}^{(j)} = \sum_{k,l} \mathrm{e}_{kl}^{(i)} \mathrm{e}_{kl}^{(j)} = \delta_{ij}, \quad \text{where } i, j = 1,2,3,4 \quad \text{and} \quad k, l = 1,2$$

By analogy to vector spaces, we can choose a different set of orthogonal basis matrices:

$$\mathbf{A} = \alpha^{(1)} \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} + \alpha^{(2)} \frac{1}{2}\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix} + \alpha^{(3)} \frac{1}{2}\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} + \alpha^{(4)} \frac{1}{2}\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$

$$= \alpha^{(1)}\mathbf{f}^{(1)} + \alpha^{(2)}\mathbf{f}^{(2)} + \alpha^{(3)}\mathbf{f}^{(3)} + \alpha^{(4)}\mathbf{f}^{(4)}$$

where $$\mathbf{f}^{(i)} \cdot \mathbf{f}^{(j)} = \sum_{k,l} \mathrm{f}_{kl}^{(i)} \mathrm{f}_{kl}^{(j)} = \delta_{ij} \quad \text{with } i, j, k, l = 1,2$$

This change of basis constitutes a rotation of the coordinate system.

In the new orthonormal set of basis matrices, we can determine the coefficients as

$$\alpha^{(i)} = \mathbf{A} \cdot \mathbf{f}^{(i)} = \sum_{k,l} A_{kl} \mathrm{f}_{kl}^{(i)} \quad \text{where } k,l = 1,2 \quad \text{and} \quad i = 1,2,3,4$$

Interpretation of the new components:

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$ Constant or irrelevant term

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}$$ Ising coupling component (the real pair interaction)

$$\begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix}$$ Magnetic field component for the Ising model

$$\begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}$$ Zero-potential component

Generalization of decomposition for symmetric $n \times n$ matrix games (now n=3)

$$\mathbf{f}^{(k,l)} = \mathbf{e}^{(k)} \otimes \mathbf{e}^{(l)} \quad \text{is the dyadic product of two vectors:} \quad \mathbf{f} = \mathbf{c} \otimes \mathbf{d} \quad \Rightarrow \quad f_{ij} = c_i d_j$$

i=(k,l), and k,l=1, …, n

$\mathbf{e}^{(k)}$ may be the traditional Cartesian basis vectors,

$$\mathbf{e}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \mathbf{e}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{e}^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \text{ then } \mathbf{f}^{(1,1)} = \mathbf{e}^{(1)} \otimes \mathbf{e}^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Or we can choose another orthogonal set of basis vectors, for example, as:

$$\mathbf{e}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \mathbf{e}^{(2)} = \begin{pmatrix} 0 \\ 1 \\ -1 \end{pmatrix}, \mathbf{e}^{(3)} = \begin{pmatrix} 2 \\ -1 \\ -1 \end{pmatrix}, \text{ then } \quad \mathbf{f}^{(1,1)} = \mathbf{e}^{(1)} \otimes \mathbf{e}^{(1)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}$$

$$\mathbf{f}^{(1,2)} = \mathbf{e}^{(1)} \otimes \mathbf{e}^{(2)} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 1 & -1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix}$$

$$\mathbf{A} = \sum \gamma_{k,l} \mathbf{f}^{(k,l)}, \text{ where } \gamma_{k,l} = \frac{\mathbf{A} \cdot \mathbf{f}^{(k,l)}}{\mathbf{f}^{(k,l)} \cdot \mathbf{f}^{(k,l)}} \qquad \cdots$$

Archetypal elementary games for 3×3 matrix games:

0) **Irrelevant term** („average payoff”): potential: $\mathbf{V}^{(av)}=0$

$$\mathbf{A}^{(av)} = a^{(av)}\mathbf{f}^{(1,1)}, \quad \mathbf{f}^{(1,1)} = \mathbf{e}^{(1)} \otimes \mathbf{e}^{(1)} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \quad \text{and} \quad a^{(av)} = \gamma_{1,1} = \frac{1}{n^2}\sum_{i,j} A_{ij}$$

1) **Games with cross-dependent payoffs**: potential: $\mathbf{V}^{(cr)}=0$

$$\mathbf{A}^{(cr)} = \gamma_{1,2}\mathbf{f}^{(1,2)} + \gamma_{1,3}\mathbf{f}^{(1,3)} = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_1 & \alpha_2 & \alpha_3 \end{pmatrix} \quad \text{and} \quad \alpha_j = \frac{1}{n}\sum_i A_{ij} - a^{(av)}$$

2) **Games with self-dependent payoffs** potential: $\mathbf{V}^{(self)}=\mathbf{A}^{(self)} + \mathbf{A}^{(self)T}$

$$\mathbf{A}^{(self)} = \gamma_{2,1}\mathbf{f}^{(2,1)} + \gamma_{3,1}\mathbf{f}^{(3,1)} = \begin{pmatrix} \beta_1 & \beta_1 & \beta_1 \\ \beta_2 & \beta_2 & \beta_2 \\ \beta_3 & \beta_3 & \beta_3 \end{pmatrix} \quad \text{and} \quad \beta_i = \frac{1}{n}\sum_j A_{ij} - a^{(av)}$$

No real player–player interactions if $\mathbf{A}=\mathbf{A}^{(cr)}+\mathbf{A}^{(self)}+\mathbf{A}^{(av)}$

All these games represent a 5-dimensional subspace, because $\sum_i \alpha_i = \sum_i \beta_i = 0.$

Archetypal elementary games for 3×3 matrix games (cont.)

The remaining dyadic products $\mathbf{f}^{(k,l)} = \mathbf{e}^{(k)} \otimes \mathbf{e}^{(l)}, \quad k,l > 1$

include transposed pairs $\mathbf{f}^{(l,k)} = \mathbf{e}^{(l)} \otimes \mathbf{e}^{(k)} = \mathbf{f}^{(k,l)\mathrm{T}}$

From these pairs we can derive **symmetric** (coordination): $\frac{1}{2}\left[\mathbf{e}^{(k)} \otimes \mathbf{e}^{(l)} + \mathbf{e}^{(l)} \otimes \mathbf{e}^{(k)}\right]$

and **antisymmetric** (cyclic) basis matrices: $\frac{1}{2}\left[\mathbf{e}^{(k)} \otimes \mathbf{e}^{(l)} - \mathbf{e}^{(l)} \otimes \mathbf{e}^{(k)}\right]$

3) **Symmetric components are composed of coordination games between strategy pairs**

$$\mathbf{A}^{(\mathrm{coor})} = \gamma_{2,2}\mathbf{f}^{(2,2)} + \gamma_{3,3}\mathbf{f}^{(3,3)} + \frac{1}{2}(\gamma_{2,3} + \gamma_{2,3})[\mathbf{f}^{(2,3)} + \mathbf{f}^{(3,2)}]$$

$$= \delta_{1,2}\begin{pmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \delta_{1,3}\begin{pmatrix} 1 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} + \delta_{2,3}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} = \begin{pmatrix} \delta_{1,2}+\delta_{1,3} & -\delta_{1,2} & -\delta_{1,3} \\ -\delta_{1,2} & \delta_{1,2}+\delta_{2,3} & -\delta_{2,3} \\ -\delta_{1,3} & -\delta_{2,3} & \delta_{1,3}+\delta_{2,3} \end{pmatrix}$$

Features: $\mathbf{A}^{(\mathrm{coor})} = \mathbf{A}^{(\mathrm{coor})\mathrm{T}}$ and the potential: $\mathbf{V}^{(\mathrm{coor})} = \mathbf{A}^{(\mathrm{coor})}$

The sum of payoffs is zero in each row and column.

(Ensuring orthogonality to the self- and cross-dependent components.)

elementary component = coordination (Ising-type interaction) between strategies *i* and *j*

there are 3 strategy pairs when *n*=3

Archetypal elementary games for 3×3 matrix games (cont.)

Antisymmetric component of the rest:

4) **Cyclic game**

Rock-paper-scissors game

$$\mathbf{A}^{(\mathrm{cycl})} = \frac{1}{2}\left[\mathbf{e}^{(2)} \otimes \mathbf{e}^{(3)} - \mathbf{e}^{(3)} \otimes \mathbf{e}^{(2)}\right] \propto \begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix}$$

Features:

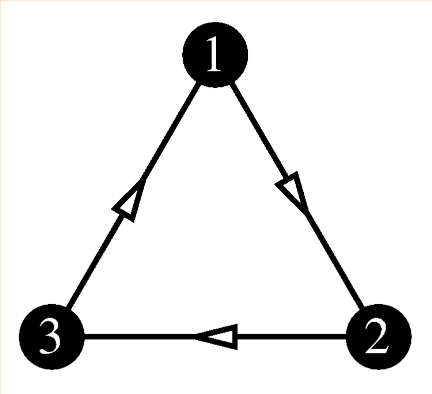
$\mathbf{A}^{(\mathrm{cycl})} = -\mathbf{A}^{(\mathrm{cycl})T}$ and it is a zero-sum game

The sum of payoffs is zero in each row and column (ensuring orthogonality)

– It does not admit a potential because along the following cycle in the strategy space the active players always increase their payoff: RP → SP → SR → PR → PS → RS → RP

– Only one mixed NE exists => coexistence of strategies (or biodiversity)

This component prevents thermodynamical behaviour
for n=3 only one cyclic component exists
The cyclic component can be considered as the adjacency matrix of
$-\mathbf{A}^{(\mathrm{cycl})}$: the direction of dominance is reversed



The whole payoff matrix can be composed as:

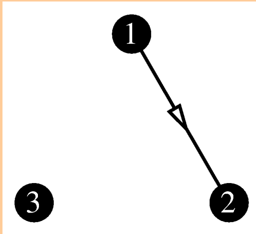
$$\mathbf{A} = \mathbf{A}^{(\mathrm{av})} + \mathbf{A}^{(\mathrm{cr})} + \mathbf{A}^{(\mathrm{self})} + \mathbf{A}^{(\mathrm{coor})} + \mathbf{A}^{(\mathrm{cycl})}$$

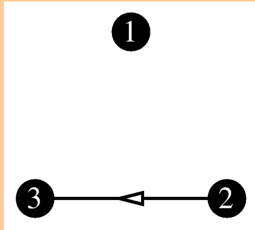
Anti-symmetric components in three-strategy games

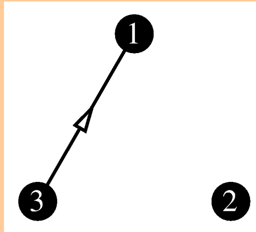
$$\frac{1}{2}\left[\mathbf{A}-\mathbf{A}^{\mathrm{T}}\right]=\begin{pmatrix} 0 & a & -c \\ -a & 0 & b \\ c & -b & 0 \end{pmatrix} = a\begin{pmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + b\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + c\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Graph representations:

Orthogonality is satisfied

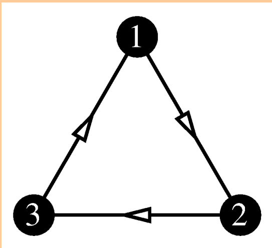


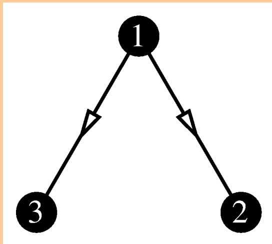


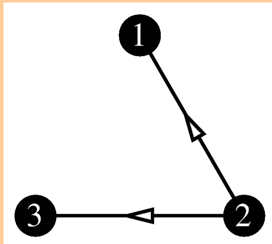


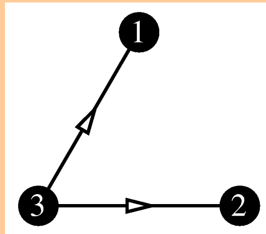
The contributions of the self- and cross-dependent components are

$$\frac{1}{2}\left[\mathbf{A}-\mathbf{A}^{\mathrm{T}}\right]=\varepsilon\begin{pmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} + h_1\begin{pmatrix} 0 & 1 & 1 \\ -1 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} + h_2\begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix} + h_3\begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & -1 \\ 1 & 1 & 0 \end{pmatrix}$$









with $h_1+h_2+h_3=0$, because the sum of the last three basis matrices is zero.

The above features remain valid for $n>3$.

Components of *n*-strategy matrix games

4+1 types of interaction:

$$\mathbf{A} = \mathbf{A}^{(\mathrm{av})} + \mathbf{A}^{(\mathrm{cr})} + \mathbf{A}^{(\mathrm{self})} + \mathbf{A}^{(\mathrm{coor})} + \mathbf{A}^{(\mathrm{cycl})}$$

$$\sum_i \alpha_i = \sum_i \beta_i = 0$$

where

$$\mathbf{A}^{(\mathrm{cr})} = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \\ \vdots & \vdots & \ddots & \vdots \\ \alpha_1 & \alpha_2 & \cdots & \alpha_n \end{pmatrix}, \quad \mathbf{A}^{(\mathrm{self})} = \begin{pmatrix} \beta_1 & \beta_1 & \cdots & \beta_1 \\ \beta_2 & \beta_2 & \cdots & \beta_2 \\ \vdots & \vdots & \ddots & \vdots \\ \beta_n & \beta_n & \cdots & \beta_n \end{pmatrix}, \quad \mathbf{A}^{(\mathrm{av})} = a^{(\mathrm{av})} \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & 1 \end{pmatrix}$$

$$\mathbf{A}^{(\mathrm{coor})} = \delta_{12} \begin{pmatrix} 1 & -1 & 0 & \cdots & 0 \\ -1 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \delta_{13} \begin{pmatrix} 1 & 0 & -1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ -1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \delta_{23} \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & -1 & \cdots & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \cdots$$

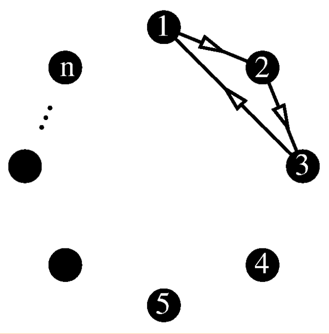
Coordination strengths ($-\delta_{12}$) are defined for each possible strategy pair [there are $n(n-1)/2$].

These latter components are generally not orthogonal to each other, but the off-diagonal entries of $\mathbf{A}^{(\mathrm{coor})}$ readily define their strengths nonetheless.

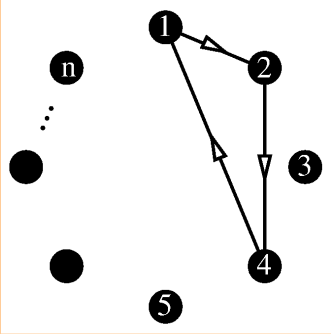
Notice: $$\mathbf{A}^{(\mathrm{coor})} = \frac{1}{2}\left[\left(\mathbf{A} - \mathbf{A}^{(\mathrm{cr})} - \mathbf{A}^{(\mathrm{self})} - \mathbf{A}^{(\mathrm{av})}\right) + \left(\mathbf{A}^{\mathrm{T}} - \mathbf{A}^{(\mathrm{cr})\mathrm{T}} - \mathbf{A}^{(\mathrm{self})\mathrm{T}} - \mathbf{A}^{(\mathrm{av})}\right)\right]$$

Independent cyclic components in n-strategy matrix games

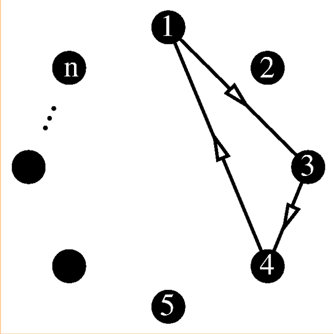
Graph representations: all the three-edge directed loops (RPS games) include strategy 1.



C(1,2,3)



C(1,2,4)



C(1,3,4)

… etc.

There are (n–1)(n–2)/2 of these RPS games.

The strength of the **C**(1,i,j) RPS component is defined by $A^{(\mathrm{cycl})}_{ij}$ because:

$$\mathbf{A}^{(\mathrm{cycl})} = \frac{1}{2}\left[\left(\mathbf{A} - \mathbf{A}^{(\mathrm{cr})} - \mathbf{A}^{(\mathrm{self})} - \mathrm{A}^{(\mathrm{av})}\right) - \left(\mathbf{A}^{\mathrm{T}} - \mathbf{A}^{(\mathrm{cr})\mathrm{T}} - \mathbf{A}^{(\mathrm{self})\mathrm{T}} - \mathbf{A}^{(\mathrm{av})}\right)\right]$$

$$= \gamma_{123}\begin{pmatrix} 0 & 1 & -1 & 0 & \cdots & 0 \\ -1 & 0 & 1 & 0 & \cdots & 0 \\ 1 & -1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \gamma_{124}\begin{pmatrix} 0 & 1 & 0 & -1 & \cdots & 0 \\ -1 & 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & 0 & \cdots & 0 \\ 1 & -1 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 0 \end{pmatrix} + \ldots$$

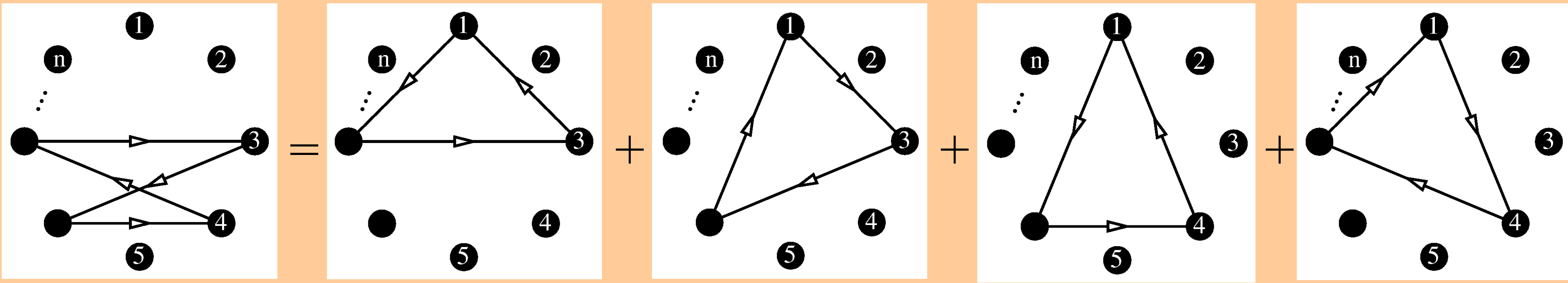
Independent cyclic components in *n*-strategy matrix games (cont.)

Evidently, we can choose other sets of independent RPS cyclic components,

for example, those including strategy 2 or strategy 3, etc.

The vanishing sum of payoffs in each row and column means that in the corresponding directed graph the number of incoming and outgoing edges are equal for each node, i.e., it is made up of directed loops.

All directed loops can be built from directed triangles as:



The game with a payoff matrix **A** is a potential game, if **A** is orthogonal to all independent cyclic components, that is,

$$\mathbf{A} \cdot \mathbf{C}(1, p, q) = \sum_{i,j=0}^{n} A_{ij} C_{ij}(1, p, q) = 0 \quad \text{for} \quad 1 < p < q$$

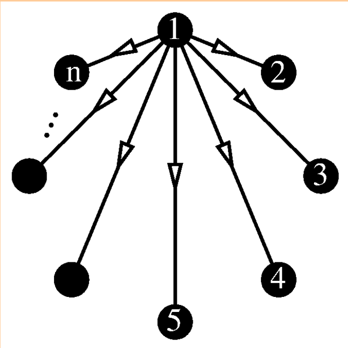
These are simple conditions for the existence of a potential and identical to those identified by Kirchhoff's law (see Lecture 6, egt06.ppt).

Independent star-like hierarchical components of n-strategy matrix games

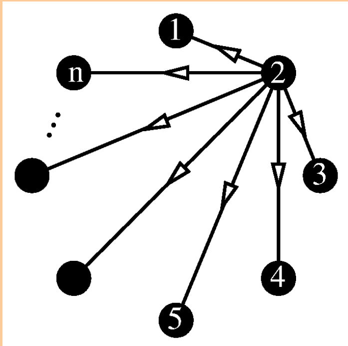
The anti-symmetric part of the potential component can be given as

$$\mathbf{A}^{(\mathrm{as})} = \frac{1}{2}\left[\left(\mathbf{A} - \mathbf{A}^{(\mathrm{cycl})}\right) - \left(\mathbf{A}^{\mathrm{T}} - \mathbf{A}^{(\mathrm{cycl})\mathrm{T}}\right)\right] = \sum_r \eta_r \mathbf{A}^{(\mathrm{h},r)}\,, \quad \eta_r = \alpha_r - \beta_r\,, \qquad \sum_r \eta_r = 0$$

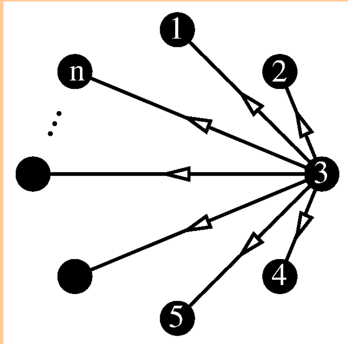
Graph representation: there are (n–1) equivalent outgoing edges from the rth strategy,



$\mathbf{A}^{(\mathrm{h},1)}$



$\mathbf{A}^{(\mathrm{h},2)}$



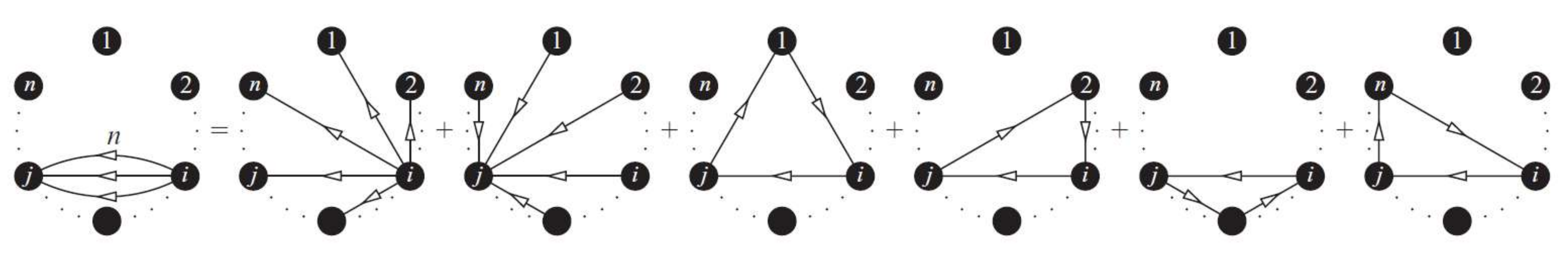
$\mathbf{A}^{(\mathrm{h},3)}$

, etc.

If $\mathbf{A}=\mathbf{A}^{(\mathrm{h},1)}$: player x using the first strategy wins 1 from player y using any other strategy.
The **payoff matrix**: **potential matrix**

$$\mathbf{A}^{(\mathrm{h},1)} = \begin{pmatrix} 0 & 1 & \cdots & 1 \\ -1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & 0 & \cdots & 0 \end{pmatrix} = \begin{pmatrix} 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}, \quad \mathbf{V}^{(\mathrm{h},1)} = \begin{pmatrix} 2 & 1 & \cdots & 1 \\ 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \cdots & 0 \end{pmatrix}$$

For directed graphs, a directed edge between the nodes *i* and *j* can be built up from the adjacency matrices of **H**(*i*), –**H**(*j*) and **C**(*i*,*j*,*k*) as



Conjecture: the concept of matrix decomposition can be applied to the quantitative analysis of directed graphs, networks, etc.

Features of star-like hierarchical components

$\mathbf{A}^{(h,1)}$ encourages both players to choose strategy 1 if $\eta_1>0$, which then yields zero payoff.

All $\mathbf{A}^{(h,r)}$ represent a similar deception, each with a different strength.

At least one of the η_r strengths is positive because

$$\sum_r \eta_r = 0.$$

In symmetric potential games (all $\eta_r=0$) the players share the income equally and there is no social dilemma.

In the presence of antisymmetric terms, a social dilemma may emerge if the symmetric and antisymmetric terms promote different Nash equilibria.

$\mathbf{A}^{(as)}$ supports the maintenance of cooperative behaviour (favouring maximum total payoff) in noisy logit dynamics if both terms favour the same Nash equilibrium, “just as Adam Smith hoped”. Otherwise, this hope it not fulfilled.

Number of orthogonal (independent) elementary games in the four types of interaction for *n* strategies

0.) Irrelevant constant: one component ($\mathbf{A}^{(av)}$)

1.) Games with cross-dependent payoffs: $n-1$ basis matrices,

2.) Games with self-dependent payoffs: $n-1$ basis matrices,

3.) Coordination games: $n(n-1)/2$ (one dimension for each possible strategy pair)

4.) Cyclic dominance: $(n-1)(n-2)/2$

equivalent to the number of independent loops deduced from Kirchhoff's law

all three-edge loops involving strategy 1 (or any other)

$n=2$: no cyclic components (all these games are potential games)

$n=3$: one rock–paper–scissors (RPS) component

$n=4$: 3 independent RPS components

Typical behaviours

Irrelevant constant

the payoff is independent of the strategy pair, players choose their strategy at random

Games with cross-dependent payoffs

the player's income is determined by their coplayer, they choose their strategy at random

Games with self-dependent payoffs:

both players act independently of each other, following their own interest undisturbed

Coordination games:

equivalent to the Ising (or Blume–Capel) model for one coordinated strategy pair sim

several coordinated components can be present simultaneously (Potts, Ashkin–Teller, etc.)

Potential games: (linear combinations of the ones above)

one or two pure and preferred Nash equilibria (in the absence of degeneracy)

homogeneous or ordered spatial strategy arrangements at low noises and

thermodynamical behaviour when the logit rule controls the evolution

more complex behaviour for imitation

Typical behaviours (cont.)

Cyclic dominance:

- prevents the existence of a potential and thermodynamical behaviour, too
- mixed Nash equilibrium (biodiversity and self-organizing patterns)
- instead of detailed balance, cyclic strategy variation can be observed

Ordinal potential games:

- the flow diagram is free of directed cycles and similar to those of potential games
- it can be considered as a potential game perturbed weakly by cyclic component(s)

Phenomena missing in physical systems:

social dilemmas

consequences of: cyclic dominance,

realistic dynamical rules,

connectivity structures,

personalities, etc.

Home assignments

7.1) Evaluate the coefficients $\alpha^{(n)}$ in the decomposition of the following antisymmetric 3x3 payoff matrix:

$$\mathbf{A}^{(\mathrm{as})} = \begin{pmatrix} 0 & a & -c \\ -a & 0 & b \\ c & -b & 0 \end{pmatrix} = \sum_{n=1}^{9} \alpha^{(n)} \mathbf{f}^{(n)} ,$$

where the 3×3 basis matrices $\mathbf{f}^{(n)}$ are those defined in the second example on page 3!

7.2) The payoff matrix of the (three-strategy) voluntary prisoner's dilemma is given by

$$\mathbf{A} = \begin{pmatrix} 0 & T & \sigma \\ S & 1 & \sigma \\ \sigma & \sigma & \sigma \end{pmatrix}$$

where $0 < \sigma < 1$ and the three strategies represent unconditional defectors, unconditional cooperators, and loners, respectively. Determine the strength of each elementary component!

Evaluate the value of σ for which a potential exists and determine the corresponding potential matrix!

7.3) The Ashkin–Teller model is a four-state version of the Ising model where the interaction between neighbouring players is defined by the payoff matrix:

$$\mathbf{A} = \begin{pmatrix} \alpha & \beta & \gamma & \delta \\ \beta & \alpha & \delta & \gamma \\ \gamma & \delta & \alpha & \beta \\ \delta & \gamma & \beta & \alpha \end{pmatrix}.$$

Determine the elementary coordination components of this matrix!