Potential games Lecture 6

When playing games, the players wish to maximize their own payoff $u_x(s_x;s_{-x})$.

For potential games, there exists a potential $V(s_1, \ldots, s_N)$ that satisfies the following condition for each player $x=1, \ldots, N$:

$$u_x(s'_x;s_{-x}) - u_x(s_x;s_{-x}) = V(s'_x;s_{-x}) - V(s_x;s_{-x}), \quad \forall\, s_x, s'_x, s_{-x}$$

where $u_x(s_x;s_{-x})$ is the payoff of player x, if she follows strategy s_x and the rest of the players choose strategy profile s_{-x}.

The potential is built up from the payoff variations caused by players modifying their strategies (sum of individual interests).

This is a strict prescription for the payoff values as the sum of payoff variations along all closed loops in the strategy profile space should be 0.

A specific and attractive feature of potential games: For suitable evolutionary dynamics, potential games evolve into the Boltzmann–Gibbs ensemble and thus the results of statistical physics (and thermodynamics) can be applied to them.

General features of potential games for two-player (matrix) games

1) The bimatrix can be mapped into a simple matrix:

$$G=(\mathbf{A},\mathbf{B})=\begin{pmatrix}(A_{11},B_{11}) & \cdots & (A_{1n},B_{1n}^{+}) \\ \vdots & \ddots & \vdots \\ (A_{n1},B_{n1}^{+}) & \cdots & (A_{nn},B_{nn})\end{pmatrix} \quad \Rightarrow \quad \mathbf{V}=\begin{pmatrix}V_{11} & \cdots & V_{1n} \\ \vdots & \ddots & \vdots \\ V_{n1} & \cdots & V_{nn}\end{pmatrix}$$

2) Homogeneity and non-uniqueness up to an arbitrary additive constant:

$$(\alpha\mathbf{A},\alpha\mathbf{B}) \quad \Rightarrow \quad \mathbf{V}=\alpha\begin{pmatrix}V_{11} & \cdots & V_{1n} \\ \vdots & \ddots & \vdots \\ V_{n1} & \cdots & V_{nn}\end{pmatrix}+\gamma\begin{pmatrix}1 & \cdots & 1 \\ \vdots & \ddots & \vdots \\ 1 & \cdots & 1\end{pmatrix}$$

3) Additivity:

$$(\mathbf{A},\mathbf{B})\Rightarrow\mathbf{V} \quad \text{and} \quad (\mathbf{A}',\mathbf{B}')\Rightarrow\mathbf{V}', \quad \text{then} \quad (\mathbf{A}+\mathbf{A}',\mathbf{B}+\mathbf{B}')\Rightarrow\mathbf{V}+\mathbf{V}'$$

Homogeneity and additivity $\Rightarrow$ linearity

4) Symmetry: $\quad$ if $\quad \mathbf{A}=\mathbf{B}, \quad$ then $\quad \mathbf{V}=\mathbf{V}^{\mathrm{T}},$

because the payoff variations of the active players are equivalent for the unilateral changes $(i,j)\rightarrow(k,j)$ and $(j,i)\rightarrow(j,k)$

General features (cont.)

5) For symmetric games (**A**=**B**) with symmetric payoff matrices ($\mathbf{A}=\mathbf{A}^T$): **V**=**A**

6) The potential is zero (**V**=0 or constant) if the payoff matrices are composed of uniform columns, that is, if

$$\mathbf{A} = \begin{pmatrix} \beta_1 & \cdots & \beta_n \\ \vdots & \ddots & \vdots \\ \beta_1 & \cdots & \beta_n \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} \gamma_1 & \cdots & \gamma_n \\ \vdots & \ddots & \vdots \\ \gamma_1 & \cdots & \gamma_n \end{pmatrix}.$$

7) If the payoff matrices are composed of rows with uniform elements as

$$\mathbf{A} = \begin{pmatrix} \beta_1 & \cdots & \beta_1 \\ \vdots & \ddots & \vdots \\ \beta_n & \cdots & \beta_n \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} \gamma_1 & \cdots & \gamma_1 \\ \vdots & \ddots & \vdots \\ \gamma_n & \cdots & \gamma_n \end{pmatrix},$$

then $\mathbf{V}=\mathbf{A}+\mathbf{B}^T$.

8) A game is a potential game if and only if all of its 2×2 subgames are potential games, too. This feature is the consequence of Kirchhoff's laws.

General features (cont.)

9) Multiagent potential games can be built up from two-person potential games, e.g.:

$$V_{\text{total}}(s) = \sum_{\langle x,y \rangle} J_{xy} V(s_x, s_y)$$

where the summation runs over $\langle xy \rangle$ interacting player pairs (J_{xy} is a coupling constant).

Each player x plays games simultaneously with several others (y). For simplicity, we assume that each player uses the same strategy in all games she participates in.

10) If the evolution of the strategy distribution is controlled by the **logit rule,** then the system evolves into the Boltzmann distribution. The logit rule consists of the following elementary steps:

- we choose a site x at random
- payoffs $u_x(s_x)$ (or potentials) are evaluated for all strategies given the current neigbourhood
- strategy s'_x is chosen with a probability (where K quantifies noise or temperature):

$$w(\mathbf{s}_x \to \mathbf{s}'_x) = \frac{\exp[u_x(s'_x)/K]}{\sum_{s''_x} \exp[u_x(s''_x)/K]} \qquad \text{(independently of } s_x\text{)}$$

General features (cont.)

11) If max(V_{ij})= V_{i*j*} then the strategy pair ($i*$,$j*$) is a preferred Nash equilibrium.

It is equivalent to the ground state in physical systems.

If $i*=j*$, then the players prefer to choose the same strategy

even for multiagent games with the same (uniform) pair interactions.

It is a unique ground state.

If $i*\neq j*$, then a chequerboard-like strategy distribution becomes a Nash equilibrium

on a square lattice and all other bipartite graphs.

It is a twofold degenerate ground state.

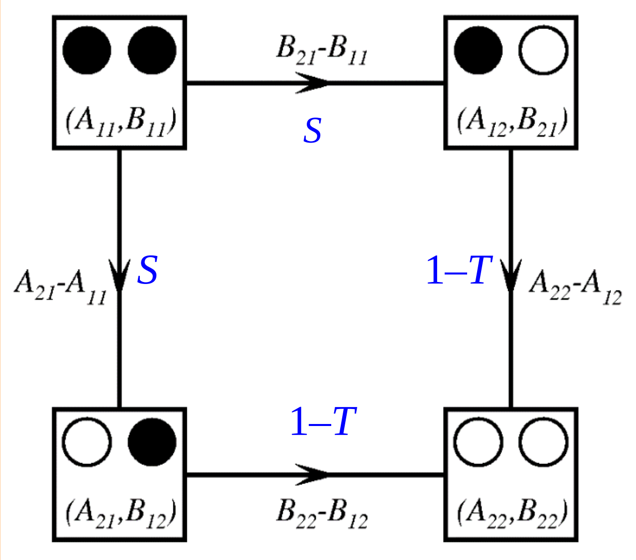
12) There are no directed loops in the flow diagram of a potential game.

In the absence of directed loops, the game has at least on pure NE. It can be proved by simple arguments based on graph theory. Namely, if we draw a directed path starting from any strategy pair (node) by following one of the outgoing edges, then we cannot arrive at a node we've visited previously (because of the absence of directed loops). After (n^2–1) steps, however, there are no more unvisited nodes left. The apparent contradiction is resolved by the existence of (at least one) pure Nash equilibrium possessing only incoming edges. This feature gives us a method to find pure NE.

All symmetric 2×2 games are potential games

Graph representation: nodes = pure strategy profiles (microscopic states)

edges = possible transitions if only one player modifies her strategy



Strategy profile: (white or black bullets),

Payoff pair: (A_{ij}, B_{ji})

Payoff variations of the player modifying her strategy:

They should add up to zero along the loop for the potential to exist, that is,

$$B_{21} - B_{11} + A_{22} - A_{12} = A_{21} - A_{11} + B_{22} - B_{12}$$

This condition is satisfied if **A**=**B**. In social dilemma notation (in blue in the dynamical graph):

$$G^{(\mathrm{sd})} = \begin{pmatrix} (0,0) & (T,S) \\ (S,T) & (1,1) \end{pmatrix}, \text{ and the potential matrix: } \quad \mathbf{V} = \begin{pmatrix} 0 & S \\ S & S+1-T \end{pmatrix}$$

Evaluation of V:

The first column and row of **V** are equivalent to the first column of **A**.

In the *n*th column the variations (payoff differences) in **V** and **A** are equivalent.

Flow graphs of symmetric 2×2 games:

Four regions:

1.) **Harmony games** (H): $T<1$, $S>0$

one pure Nash equilibrium: *CC* (no problems)

2.) **Hawk-dove games** (HD): $T>1$, $S>0$

two equivalent pure Nash equilibria: *CD* and *DC*

3.) **Stag hunt games** (SH): $T<1$, $S<0$

preferred Nash equilibrium: *DD* if $S-T+1<0$

CC if $S-T+1>0$

4.) **Prisoner's dilemmas** (PD): $T>1$, $S<0$

one preferred Nash equilibrium: *DD*

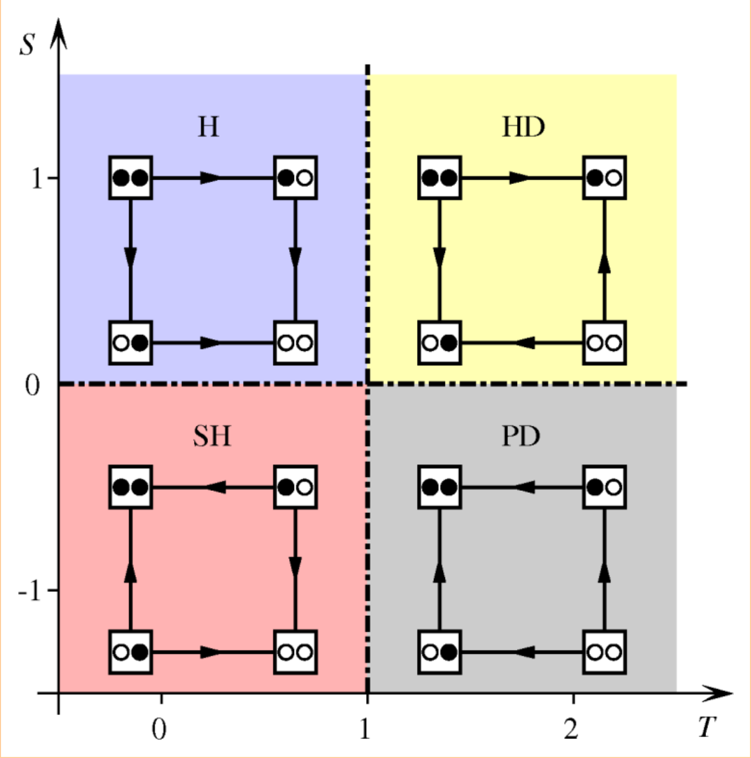
representing a **tragedy of the community**

strategies:

C: cooperation

D: defection

Only four types of flow graph exist!



payoffs:

R: Reward for mutual cooperation =1

P: Punishment for mutual defection =0

T: Temptation to choose defection

S: Sucker's payoff

Non-symmetric 2×2 potential games

An extension of social dilemmas by introducing three payoff parameters (*a*, *b*, and *c*):

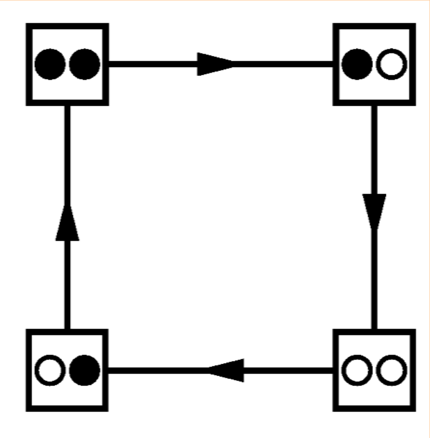
$$G=\begin{pmatrix}(0,0) & (T,S+a)\\ (S,T+b) & (1,1-c)\end{pmatrix}\rightarrow\quad \mathbf{V}=\begin{pmatrix}0 & S+a\\ S & S+1-T+a\end{pmatrix}\qquad \text{if}\ \ a+b+c=0$$

A potential exists with two tunable payoff parameters (e.g., *a* and *b*).

Counterexample (**matching pennies**):

$$G^{(\mathrm{mp})}=\begin{pmatrix}(1,-1) & (-1,1)\\ (-1,1) & (1,-1)\end{pmatrix}$$

Flow diagram:



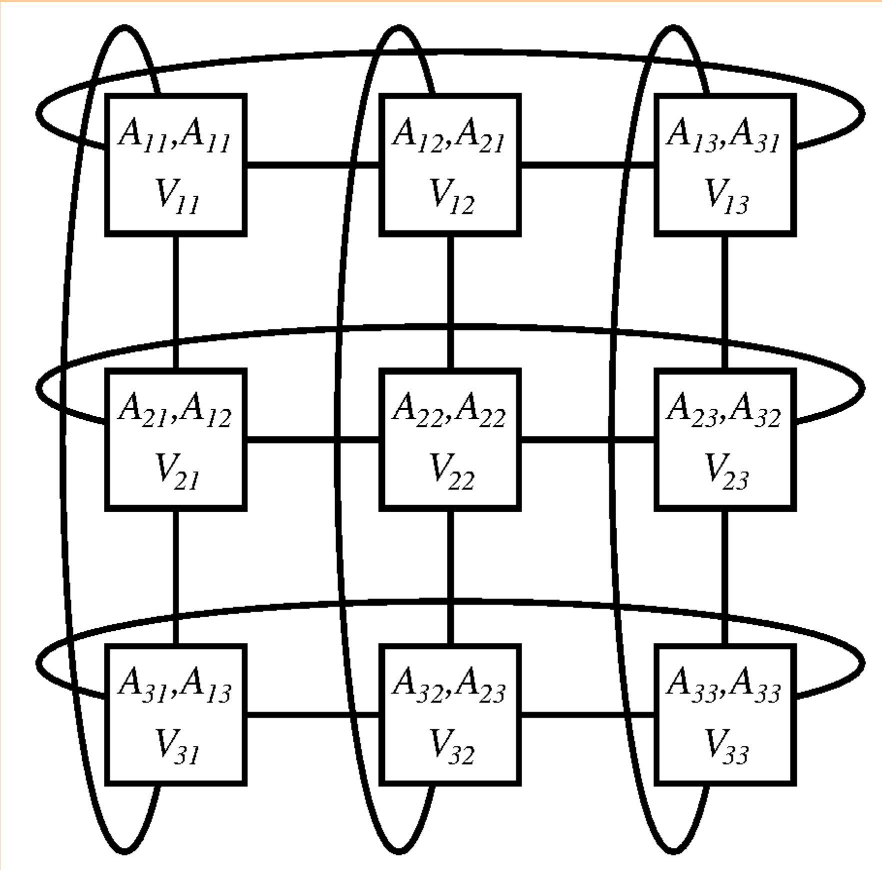
If $(a+b+c)\neq 0$, then

$$G=G^{(\mathrm{pot})}+\varepsilon G^{(\mathrm{mp})}\quad \text{where}\quad \varepsilon=\frac{1}{8}(a+b+c)$$

„Matching pennies" represents an interaction that generates cyclic consecutive strategy changes.

Two-person *n*-strategy symmetric potential games

Dynamical graph for n=3:



The game admits a potential if and only if the sums of individual incentives are zero along all possible loops.

It is enough to study the N_i independent loops.

This dynamical graph has:

9 nodes

N_e=18 edges

N_s=8 edges in its spanning tree

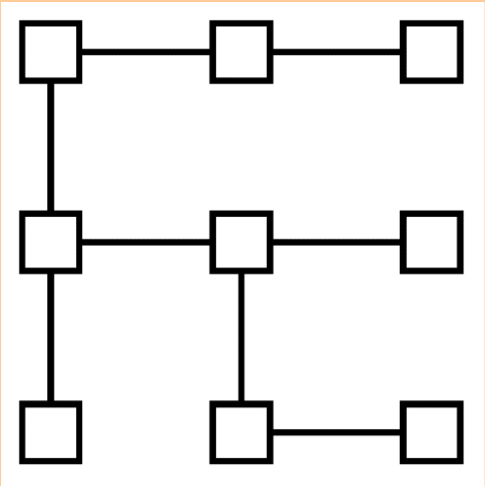
Kirchhoff's laws: ($N_i = N_e - N_s$)

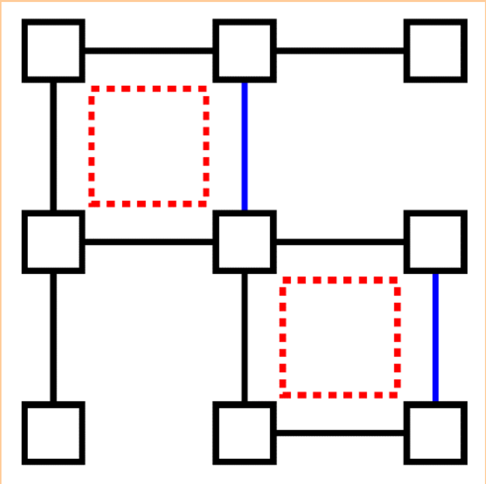
N_i=18–8=10 independent loops

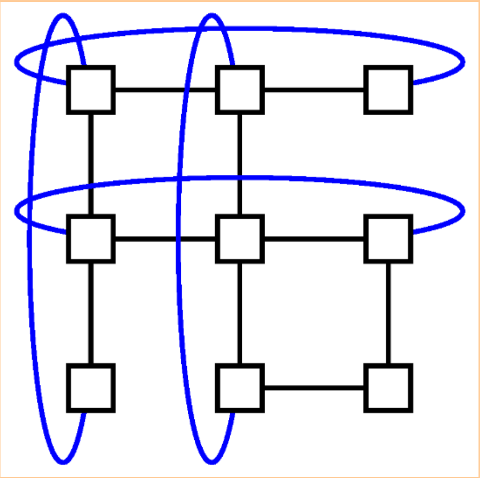
Some loops trivially satisfy the zero-sum condition:

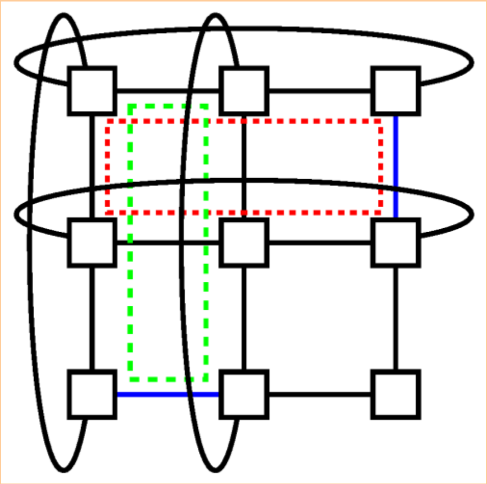
- 2×2 symmetric subgames (with both players using the same strategy pair; there are 3 for n=3)
- independent loops of nodes in a single row or column (6 for n=3)
- Consequently, there is only **1** relevant loop we need to consider when n=3.

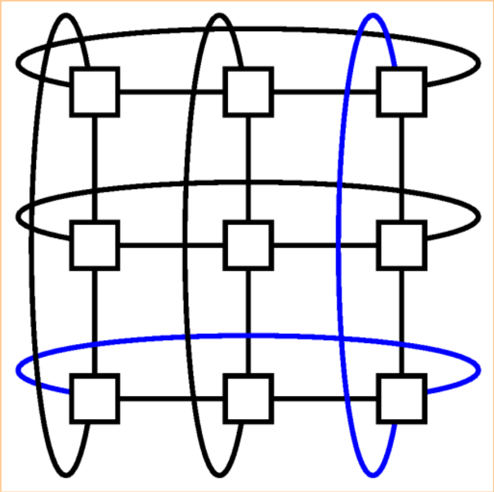
Graph representation of identifying the relevant independent loop for n=3











Take the spanning tree.

Add 10 edges to complete the loops.

8 edges can be related to trivial solutions.

The 2 remaining nontrivial loops (red and green dashed lines) give equivalent conditions.

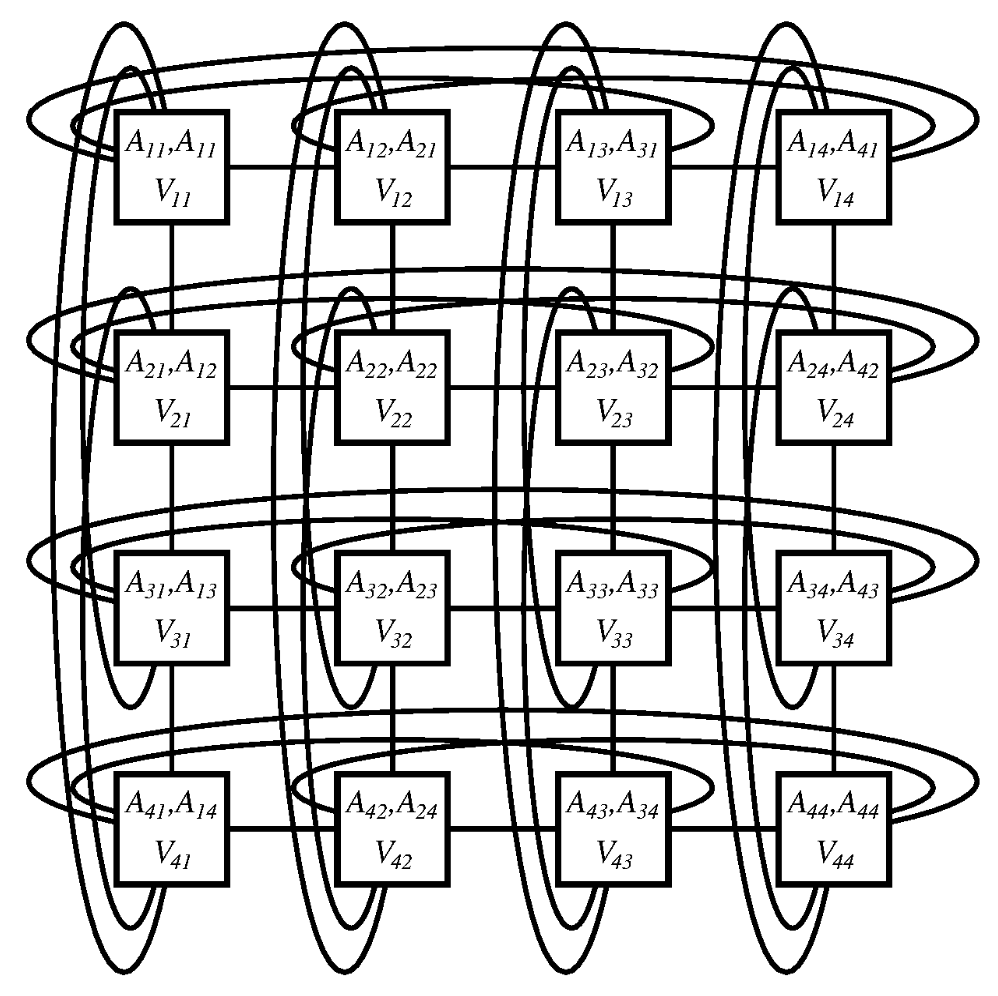
One relevant loop [(1,1)→(1,3)→(2,3)→(2,1)→(1,1)] gives the condition

$$A_{31}-A_{11}+A_{23}-A_{13}+A_{12}-A_{32}+A_{11}-A_{21}=0, \text{ that is, } \quad A_{12}-A_{21}+A_{23}-A_{32}+A_{31}-A_{13}=0.$$

The other loop [(1,1)→(3,1)→(3,2)→(1,2)→(1,1)] leads to the same condition.

Two-person *n*-strategy symmetric potential games (generalization to $n>3$)

Dynamical graph for $n=4$.



Kirchhoff's laws (for $n>3$):

The number of:

nodes: n^2

edges: $(n-1)n^2$

edges in spanning tree: $(n+1)(n-1)$

independent loops:

$$N_i=(n-1)n^2-(n+1)(n-1)=n^3-2n^2+1$$

2×2 symmetric subgames:

$$n(n-1)/2$$

Independent vertical and horizontal loops:

$$2n\left(\frac{n(n-1)}{2}-(n-1)\right)=n^3-3n^2+2n$$

Finally, we have N_r relevant loops to actually check:

$$N_r=N_i-\frac{n(n-1)}{2}-(n^3-3n^2+2n)=\frac{(n-1)(n-2)}{2}=\binom{n-1}{2}$$

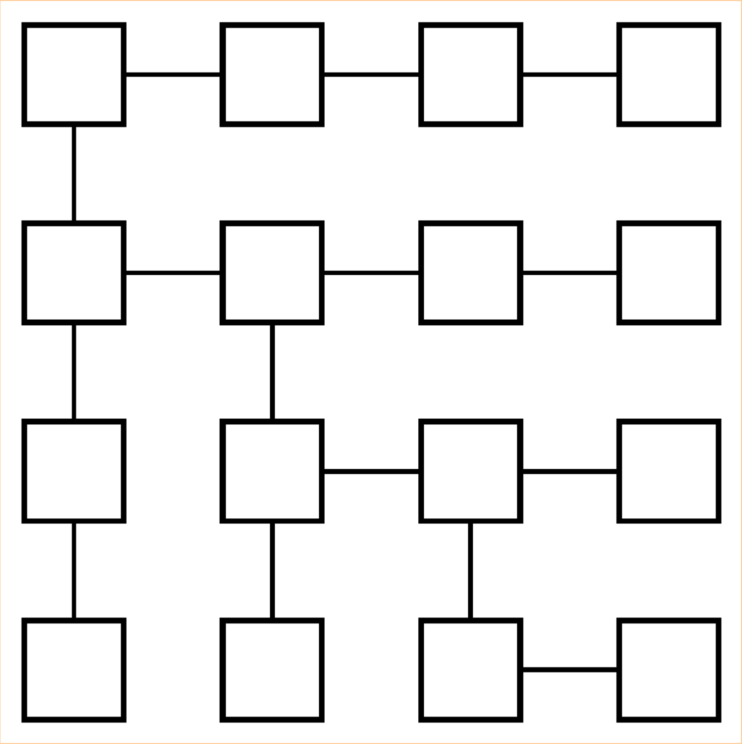
Iterative algorithm to derive relevant loops for the (*n*+1)-strategy case

assuming knowledge of the *n*-strategy case

Example for $n=3$

Step 1: Derive the $n+1$ spanning tree.

This tree contains the one for *n*.



Iterative algorithm to derive relevant loops for the (n+1)**-strategy case** (cont.)

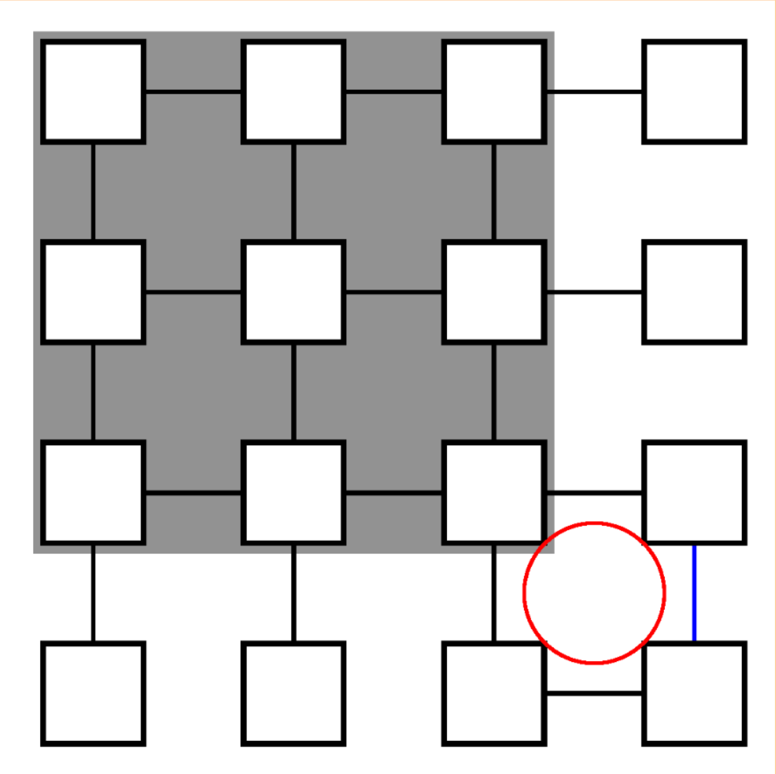
Step 2:

Add all edges that belong to the n-strategy subgame.

This process involves all relevant loops for n.

Step 3: Add the missing edge of the red loop.

The resulting graph is symmetric.



Iterative algorithm to derive relevant loops for the (n+1)**-strategy case** (cont.)

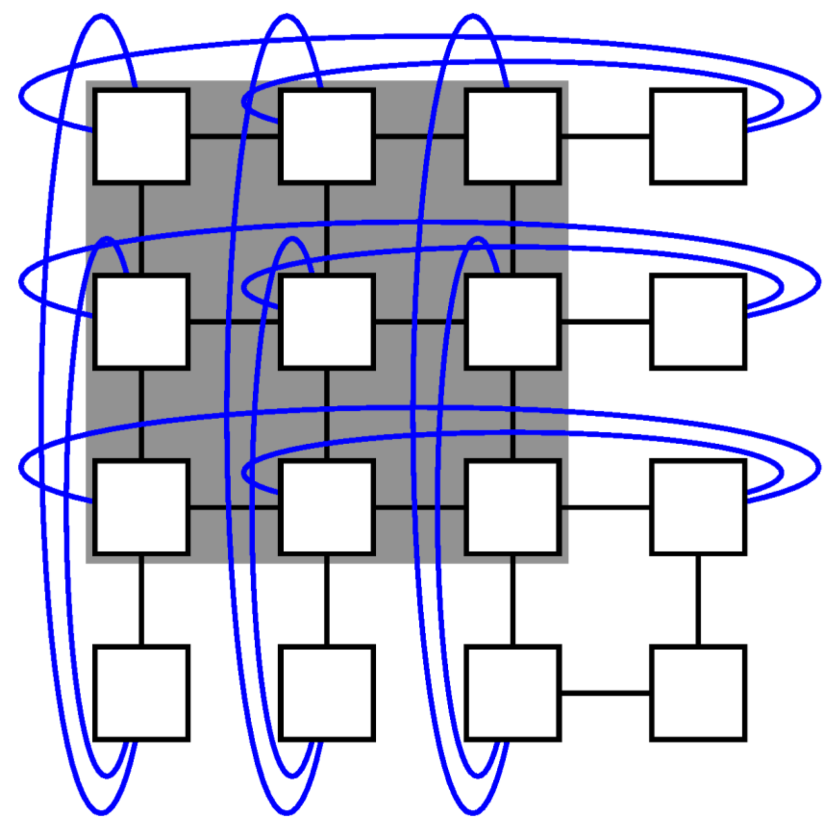
Step 4:

Add all missing horizontal edges in the first n rows.

Step 5:

Add all missing vertical egdes in the first n columns.

So far, only (trivial) zero-sum loops were created.

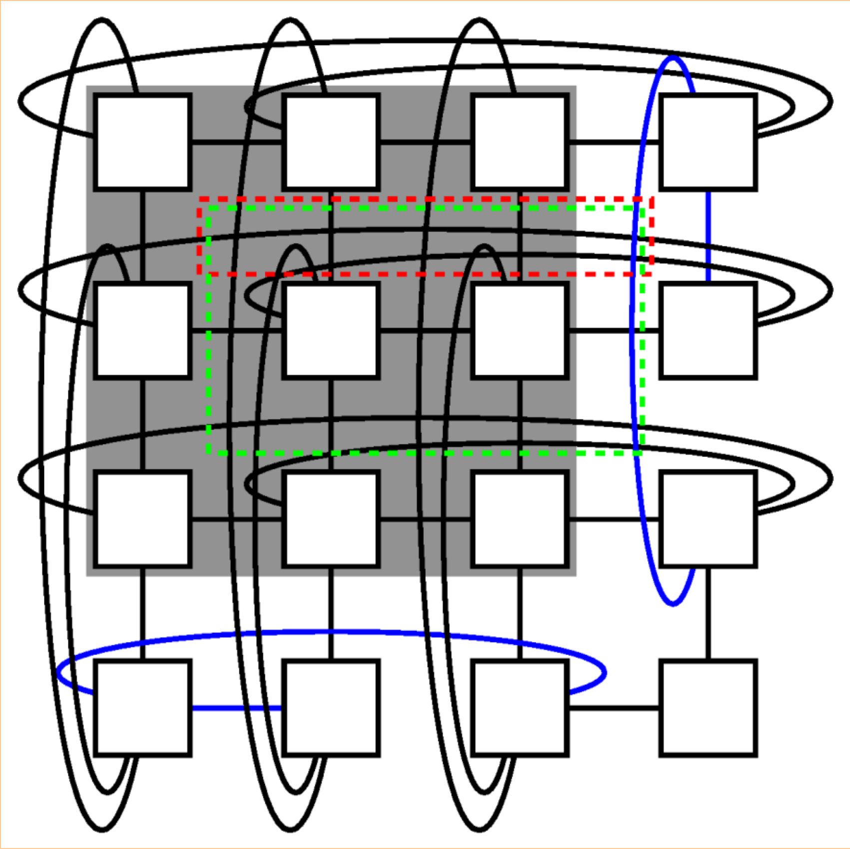


Iterative algorithm to derive relevant loops for the (n+1)**-strategy case** (cont.)

Step 6:

Add (n–1) vertical edges in the nth column, closing the red and green loops.

These steps give us (n–1) relevant loops.



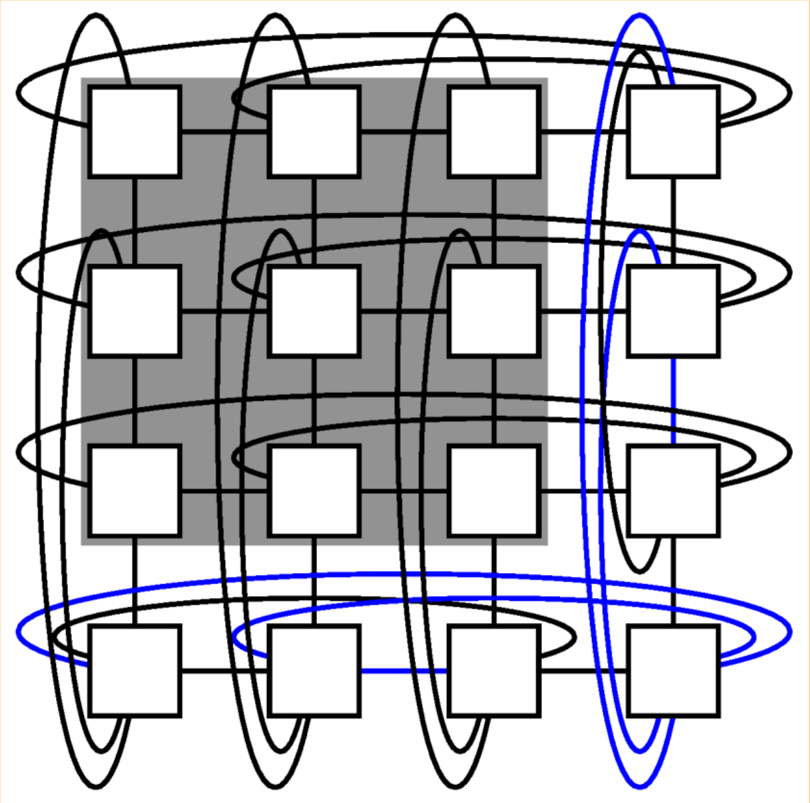
Step 7: Repeat Step 6, but horizontally.

This gives no new relevant loops,

because of the reflection symmetry.

These loops are not shown here.

Iterative algorithm to derive relevant loops for the (n+1)**-strategy case** (cont.)



Step 8:

Add all missing horizontal and vertical egdes in the nth row and column.

These steps yield 2(n–1) [here 6] trivial loops.

Summary:

This algorithm provides a total of (n–1)(n–2)/2 relevant loops for the n-strategy game that have a common feature:

$(1,1)\rightarrow(1,i)\rightarrow(j,i)\rightarrow(j,1)\rightarrow(1,1)$ with $1 < j < i \leq n$ with conditions:

$$A_{1j}-A_{j1}+A_{ji}-A_{ij}+A_{i1}-A_{1i}=0$$

These conditions are equivalent to orthogonality to the rock–paper–scissors components to be discussed in the next lecture.

Home assignments:

6.1. Prove statements 6) and 7) from page 3!

6.2. The voluntary version of the Prisoner's dilemma game has three strategies labeled *D*, *C*, and *L* (*L* as in loner) and its payoff matrix can be given as:

$$\mathbf{A} = \mathbf{B} = \begin{pmatrix} 0 & T & \sigma \\ S & 1 & \sigma \\ \sigma & \sigma & \sigma \end{pmatrix}.$$

Is it a potential game?

6.3. What payoff matrix describes a symmetric 2×2 game extended by a third strategy that mixes its first and second strategies with probabilities q and $(1–q)$? Use the P=0, R=1, T, S parametrization! Prove that the resulting three-strategy game is a potential game!