Matrix games

Two players (x and y), who each choose one of their strategies simultaneously.

n and m options (henceforth n=m) denoted by unit vectors:

$$\mathbf{s}_x, \mathbf{s}_y = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \quad \cdots, \quad \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \qquad \text{(pure strategies)}$$

Payoff matrix = tabulation of payoffs

$$\mathbf{A} = \begin{pmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & A_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{pmatrix} \quad \text{and} \quad \mathbf{B} = \begin{pmatrix} B_{11} & B_{12} & \cdots & B_{1n} \\ B_{21} & B_{22} & \cdots & \vdots \\ \vdots & \vdots & \ddots & \vdots \\ B_{n1} & B_{n2} & \cdots & B_{nn} \end{pmatrix}$$

Payoffs: $U_x = \mathbf{s}_x \cdot \mathbf{A}\mathbf{s}_y$ and $U_y = \mathbf{s}_y \cdot \mathbf{B}\mathbf{s}_x$ or $U_y = \mathbf{s}_x \cdot \mathbf{B}^{\mathrm{T}}\mathbf{s}_y$

Strategy profile: $(\mathbf{s}_x, \mathbf{s}_y)$

Mixed strategies:

Generalization of pure strategies for both players

$$\mathbf{s}_x = \begin{pmatrix} e_{x1} \\ e_{x2} \\ \vdots \\ e_{xn} \end{pmatrix}, \quad 0 \le e_{x1}, e_{x2}, \cdots, e_{xn} \le 1 \text{ and } \sum_j e_{xj} = 1 \qquad (n-1)\text{-simplex}$$

Dimension of simplex: $(n–1)$

Interpretations:

(i) player x chooses her j^{th} strategy with a probability e_{xj}

(ii) Players x and y represent large populations of players (with $N\to\infty$)
each participant in team x plays against all the participants in team y

Average payoffs can be given as:

$$U_x = \mathbf{s}_x \cdot \mathbf{A}\mathbf{s}_y \quad \text{and} \quad U_y = \mathbf{s}_y \cdot \mathbf{B}\mathbf{s}_x$$

or:

$$U_x = \sum_{i,j=1}^{n} e_{xi} A_{ij} e_{yj}$$

Bimatrix formalism

The game is defined by two matrices and denoted as: G=(**A**,**B**^{T})

in the three-strategy case

$$G = \begin{pmatrix} (A_{11}, B_{11}) & (A_{12}, B_{21}) & (A_{13}, B_{31}) \\ (A_{21}, B_{12}) & (A_{22}, B_{22}) & (A_{23}, B_{32}) \\ (A_{31}, B_{13}) & (A_{32}, B_{23}) & (A_{33}, B_{33}) \end{pmatrix}.$$

(**B** is transposed)

The large number of parameters (payoffs) causes difficulties in the classification of games and in the systematic investigation of features

Symmetries:

The game is called symmetric if **A**=**B**

the players are equivalent: they exchange payoffs when they exchange strategies

The payoff matrix itself can also be symmetric: **A**=**A**^{T}

these games belong to the class of potential games

Zero-sum games: **B**^{T}=–**A**

Game theoretical solutions

Minimax strategy for zero-sum (or constant-sum) games

Neumann: Instead of selecting a strategy with the maximum payoff we should choose a strategy that provides the highest minimum payoff (**minimax strategy**). Accordingly, we determine our minimum payoff for each strategy and we select the one where this minimum is the highest.

Example:

$$G = \begin{pmatrix} (0,0) & (1,-1) & (-2,2) \\ (1,-1) & (3,-3) & (4,-4) \\ (2,-2) & (-5,5) & (5,-5) \end{pmatrix}, \quad \text{or}: \quad \mathbf{A} = \begin{pmatrix} 0 & 1 & -2 \\ 1 & 3 & 4 \\ 2 & -5 & 5 \end{pmatrix}$$

Solution: minimum payoffs for the first player

$$G = \begin{pmatrix} (0,0) & (1,-1) & (-2,2) \\ (1,-1) & (3,-3) & (4,-4) \\ (2,-2) & (-5,5) & (5,-5) \end{pmatrix}, \qquad \begin{matrix} -2 \\ 1 \\ -5 \end{matrix}$$

The suggested strategy pair: (2,1)

resulting payoffs: (1,-1)

Dominated strategies:

Strategy $\mathbf{s}_x^{(-)}$ is strictly dominated by strategy $\mathbf{s}_x^{(+)}$, if for any $\mathbf{s}_\mathrm{y}$

$$\mathbf{s}_\mathrm{x}^{(-)} \cdot \mathbf{A}_x \mathbf{s}_y < \mathbf{s}_x^{(+)} \cdot \mathbf{A}_x \mathbf{s}_y \,.$$

Strategy $\mathbf{s}_x^{(-)}$ is non-strictly dominated, if

$$\mathbf{s}_x^{(-)} \cdot \mathbf{A}_x \mathbf{s}_y \leq \mathbf{s}_x^{(+)} \cdot \mathbf{A}_x \mathbf{s}_y$$

Rational players do not use these strategies. This fact is known by the others, too. Consequently, these options can be neglected by both players.

Consecutive elimination of strictly dominated strategies can lead to a unique solution.

A strategy profile is Pareto optimal if there are no other strategy profiles that provide some of the players a higher income without a cost to others.

There are games with more than one Pareto optimal strategy
(muddying the concept of optimality)

Nash equilibrium

Strategy profile $(\mathbf{s}^*_x, \mathbf{s}^*_y)$ is a Nash equilibrium (NE) if

$$U_x^* = \mathbf{s}_x^* \cdot \mathbf{A}\mathbf{s}_y^* \geq \mathbf{s}_x \cdot \mathbf{A}\mathbf{s}_y^* \quad \text{for all} \quad \mathbf{s}_x \neq \mathbf{s}_x^* \quad \text{and}$$

$$U_y^* = \mathbf{s}_y^* \cdot \mathbf{B}\mathbf{s}_x^* \geq \mathbf{s}_y \cdot \mathbf{B}\mathbf{s}_x^* \quad \text{for all} \quad \mathbf{s}_y \neq \mathbf{s}_y^*$$

For strict NE:

$$U_x^* = \mathbf{s}_x^* \cdot \mathbf{A}\mathbf{s}_y^* > \mathbf{s}_x \cdot \mathbf{A}\mathbf{s}_y^* \quad \text{for all} \quad \mathbf{s}_x \neq \mathbf{s}_x^* \quad \text{and}$$

$$U_y^* = \mathbf{s}_y^* \cdot \mathbf{B}\mathbf{s}_x^* > \mathbf{s}_y \cdot \mathbf{B}\mathbf{s}_x^* \quad \text{for all} \quad \mathbf{s}_y \neq \mathbf{s}_y^*$$

Both players are satisfied because they cannot increase their own payoff by unilateral strategy modification. They feel that they have achieved the best result under the given conditions.

Game theory suggests that the players choose Nash equilibria.

Normal-form game:

N players (x=1, 2, …,N), each with a finite number of pure strategies.

payoffs are tabulated for each player: $u_x(\mathbf{s}_1, \ldots, \mathbf{s}_N)$

Two-person matrix games are normal-form games.

NE can be defined as for two-player games:

Unilateral deviation from $(\mathbf{s}^*_1, \ldots, \mathbf{s}^*_N)$ is not beneficial to any of the players.

Nash theorem: for normal-form games there exists at least one NE, possibly involving mixed strategies.

Consequence: A solution (a game theoretical suggestion) always exists.

Problem: Typically, many NE exist.

So we need further criteria to select/prefer one of them:

- select a Pareto optimal NE (Payoff Dominance)

  (then nobody can choose a better paying NE without decreasing another's income)

- reduce risk (called Risk Dominance introduced by Harsányi and Selten)

  choose one that provides the highest income against „random" players

- maximize total income,

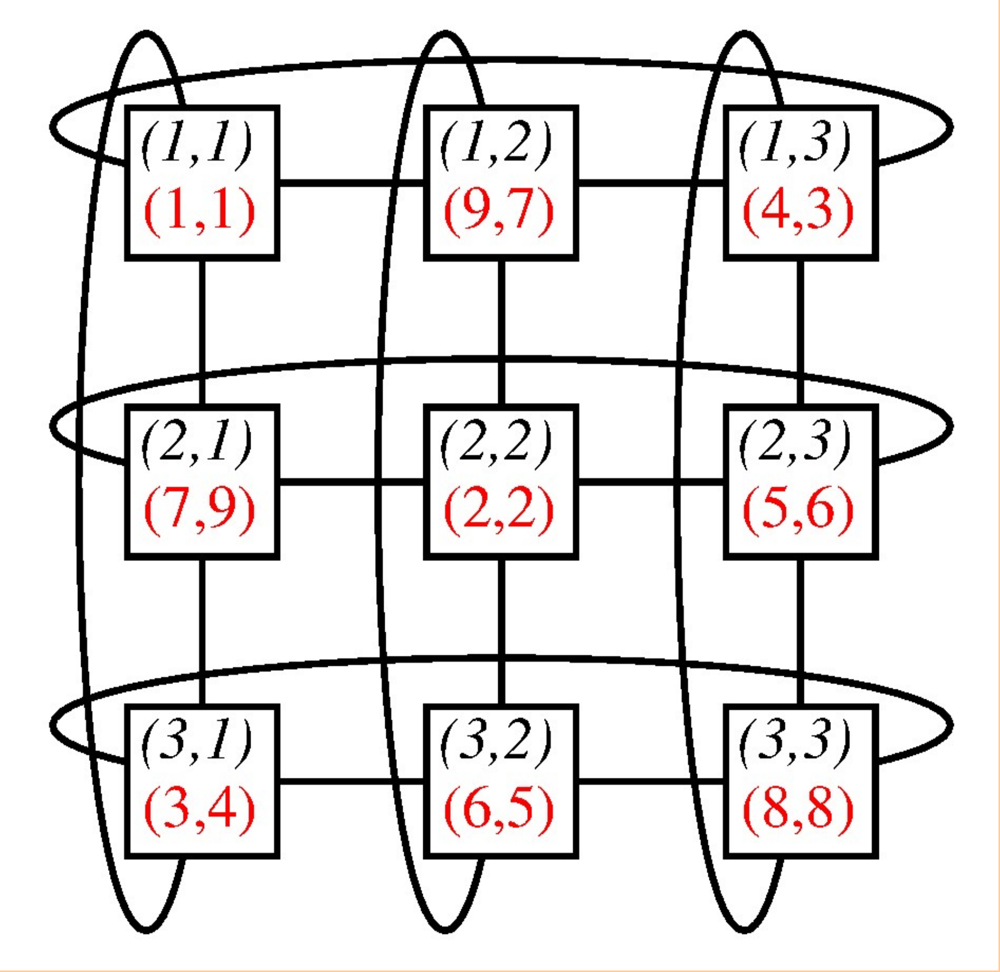
- communication, trust and agreement, …

Dynamical graph Schnakenberg, *Rev. Mod. Phys.* 48 (1976) 571

It is an old concept in statistical physics.

The nodes of the graph (here boxes) represent the microscopic states (here strategy profiles) of the system. The edges correspond to unilateral strategy changes.

For example, in a two-player three-strategy matrix game:



The upper pair of numbers indicates strategy labels, the red (lower) pairs of numbers define payoffs.

Along the edges, we can use arrows to indicate the preferred strategy, the one that provides a higher income for the active player.

→ **Flow graph**

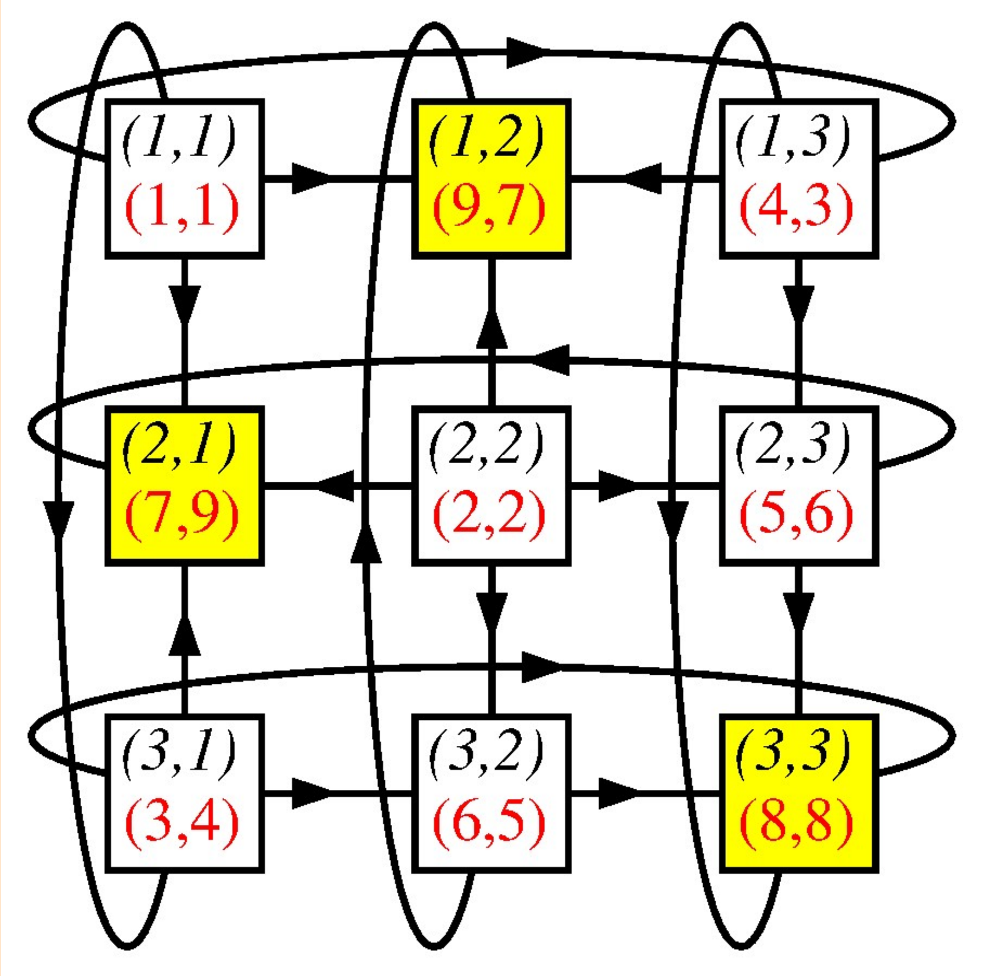
It suffices to know the rank of payoffs.

Flow graph

Directed graph

Pure and strict Nash equilibrium (NE): nodes with only incoming edges

In the previous 3×3 example: (the yellow boxes are NE)



It is a simple method for finding NE.

Here the integers reflect payoff ranks.

Notice that at most one pure NE can exist in each row and column.

There are games without pure NE.

Evaluation of mixed NE for the hawk–dove game

Two strategies: hawk = aggressive

dove = conflict-avoiding

Payoff matrix:

$$\mathbf{A} = \begin{array}{c} \\ \text{hawk} \\ \text{dove} \end{array} \begin{array}{c} \begin{array}{cc} \text{hawk} & \text{dove} \end{array} \\ \begin{pmatrix} (v-c)/2 & v \\ 0 & v/2 \end{pmatrix} \end{array} \quad , \quad c > v > 0$$

if players x and y use mixed strategies: $\mathbf{s}_x = \begin{pmatrix} p \\ 1-p \end{pmatrix}$ and $\mathbf{s}_y = \begin{pmatrix} q \\ 1-q \end{pmatrix}$

Payoff of player x:

$$U_x = \begin{pmatrix} p & (1-p) \end{pmatrix} \begin{pmatrix} (v-c)/2 & v \\ 0 & v/2 \end{pmatrix} \begin{pmatrix} q \\ 1-q \end{pmatrix}$$

$$= \begin{pmatrix} p & (1-p) \end{pmatrix} \begin{pmatrix} (v-c)q/2 + v(1-q) \\ v(1-q)/2 \end{pmatrix}$$

$$= (v-c)\,pq/2 + v(1-q)\,p + v(1-q)(1-p)/2$$

Similarly:

$$U_y = (v-c)qp/2 + v(1-p)q + v(1-p)(1-q)/2$$

Players receive the highest payoff when ($0<p,q<1$) and

$$\frac{\partial U_x}{\partial p} = 0 \quad \text{and} \quad \frac{\partial U_y}{\partial q} = 0,$$

that gives:

$$0 = (v-c)q/2 + v(1-q) - v(1-q)/2$$
$$0 = (v-c)p/2 + v(1-p) - v(1-p)/2$$

multiplying by 2:

$$0 = (v-c)q + v(1-q) = v - cq$$
$$0 = (v-c)p + v(1-p) = v - cp$$

Finally:

$$p = q = \frac{v}{c}$$

The three Nash equilibria:

$$\mathbf{s}^*_{x1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \mathbf{s}^*_{y1} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}; \quad \mathbf{s}^*_{x2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \mathbf{s}^*_{y1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}; \quad \mathbf{s}^*_{x3} = \mathbf{s}^*_{y3} = \begin{pmatrix} v/c \\ 1 - v/c \end{pmatrix}.$$

Determination of mixed NE with calculus of variations (for symmetric games)

$$\mathbf{s}_x^+ = (x_1, \cdots, x_n) \quad \text{and} \quad \mathbf{s}_y^+ = (y_1, \cdots, y_n); \quad 0 < x_i, y_i < 1 \quad \forall i$$

$$\sum_{i=1}^{n} x_i = 1 \quad \text{and} \quad \sum_{i=1}^{n} y_i = 1 .$$

Players x and y maximize their payoff by choosing optimal x_i and y_i values under the ‘normalization’ condition.

Payoffs:

$$U_x = \sum_{i,j} x_i A_{ij} y_j \quad \text{and} \quad U_y = \sum_{i,j} y_i A_{ij} x_j$$

Optimize the functions

$$F_x = \sum_{i,j} x_i A_{ij} y_j + \lambda_x \sum_i x_i$$

$$F_y = \sum_{i,j} y_i A_{ij} x_j + \lambda_y \sum_i y_i$$

where the normalization conditions are taken into consideration via the Lagrange multipliers λ_x and λ_y.

Accordingly, $$\frac{\partial F_x}{\partial x_i} = 0 \quad \text{and} \quad \frac{\partial F_y}{\partial y_i} = 0; \quad i = 1, \cdots, n$$

Evaluating the derivatives gives:

$$0 = \sum_j A_{ij} y_j + \lambda_x \quad \text{and} \quad 0 = \sum_j A_{ij} x_j + \lambda_y \quad \forall i,$$

which can be written in matrix/vector notation as:

$$\mathbf{A}\mathbf{s}_y^* = -\lambda_x \mathbf{1}, \quad \text{and} \quad \mathbf{A}\mathbf{s}_x^* = -\lambda_y \mathbf{1}, \quad \text{where} \quad \mathbf{1} = \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix}.$$

Multiplication by $\mathbf{A}^{-1}$ from the left yields

$$\mathbf{s}_x^* = \mathbf{s}_y^* = \frac{1}{\Omega} \mathbf{A}^{-1} \mathbf{1}, \quad \text{where} \quad \det \mathbf{A} \neq 0 \quad \text{is assumed}$$

$\lambda_x = \lambda_y = 1/\Omega$ (Ω is a prefactor for normalization).

We should check the feasibility of the values of x_i and y_i and whether the result is actually a maximum or a minimum.

Examples of two-person games in bimatrix formalism

1. **Coordination games**:

The players have to choose between two options, identical choices are favoured

Real life situations:

- left-hand or right-hand traffic
- metric or imperial units (metres or inches)
- Linux or Windows
- which mailing system to use
- which technology to use

Agreements or rules can bypass the decision.

Tabulated payoffs (in bimatrix form):

$$G = \begin{pmatrix} (1,1) & (0,0) \\ (0,0) & (1,1) \end{pmatrix}$$

It is symmetric (and a potential game)

resembles particle-particle interactions

NE: (1,1) or (2,2) strategy pairs

+ a mixed NE that is not an evolutionarily stable strategy (not an ESS)

Other coordination games

1b. **Battle of the sexes**

Wife and husband, who have forgotten whether they have agreed to go to a ballet performance or a football match tonight, have to decide where to go without talking to each other.

Payoffs:

$$G = \begin{pmatrix} (2,1) & (0,0) \\ (0,0) & (1,3) \end{pmatrix}$$

Not symmetric!

1c. **Stag hunt game**

Story: Rousseau: A Discourse on Inequality

„If it was a matter of hunting a deer, everyone well realized that he must remain faithfully at his post; but if a hare happened to pass within the reach of one of them, we cannot doubt that he would have gone off in pursuit of it without scruple and, having caught his own prey, he would have cared very little about having caused his companions to lose theirs."

Strategy 1: to hunt a hare;

Strategy 2: to hunt a stag

The payoffs:

$$G = \begin{pmatrix} (1,1) & (2,0) \\ (0,2) & (3,3) \end{pmatrix}$$

2. **Anticoordination game**

Real life situation: Two people arrive at a door at the same time. Each player should choose whether to walk through or let the other pass. Choosing opposites is preferred. The choice between the two equivalent possibilities can be simplified by customs or rules decided beforehand.

Payoffs:

$$G = \begin{pmatrix} (0,0) & (1,1) \\ (1,1) & (0,0) \end{pmatrix}$$

symmetrical

Two pure (Pareto optimal) NE

+ a mixed NE (which is an ESS)

2b. The **Chicken game** is 'played' on a long straight road by reckless young delinquents who drive fast towards each other on a collision course. As they approach each other they have two choices: (i) keep going straight ahead or (ii) avoid the accident. If one of them ('the chicken') swerves before the other, then he will be shamed for his cowardice and the winner gets bragging rights.

2c. **Hawk–dove game** (introduced by Maynard Smith and used widely in biology)

In this game the players compete for some resource and can either take an aggressive (hawk) or a cooperative and conflict-avoding (dove) approach. When two doves meet, they share the resource equally. When two hawks meet, they engage in a fight and get seriously injured. If a hawk meets a dove, the hawk takes the whole resource.

Typical payoffs in bimatrix form:

$$G = \begin{pmatrix} (-10,-10) & (2,0) \\ (0,2) & (1,1) \end{pmatrix}$$

2d. In the **Snowdrift game** drivers are trapped on opposite sides of a snowdrift (e.g., in Switzerland). They must choose between (i) getting out and shoveling or (ii) remaining in the car. If both are willing to shovel, then they will arrive home earlier. It is more convenient, however, to wait in the car for the other to remove the snowdrift. They will both remain stuck in their cars for a long time if they both choose the second option.

3. **Prisoner's dilemma**

Two burglars are arrested after their joint burglary and held separately by the police. In the absence of sufficient proof to have them convicted, the prosecutor offers them the same deal: (i) If one confesses (called defection) and the other remains silent (called cooperation), then the silent accomplice receives a five-month sentence and the confessor walks free. (ii) If both stay silent, then they become free after one month in the absence of proof. (iii) If both confess, then each burglar has to serve a three-month sentence.

The quantified payoffs are defined by the time (in months) they stay free in comparison to the maximal punishment:

$$G = \begin{array}{c} \\ D \\ C \end{array} \begin{array}{c} \begin{array}{cc} D & C \end{array} \\ \begin{pmatrix} (-3,-3) & (0,-5) \\ (-5,0) & (-1,-1) \end{pmatrix} \end{array} \rightarrow G = \begin{array}{c} \\ D \\ C \end{array} \begin{array}{c} \begin{array}{cc} D & C \end{array} \\ \begin{pmatrix} (2,2) & (5,0) \\ (0,5) & (4,4) \end{pmatrix} \end{array}$$

The best (rational) choice for both players is to choose defection independent of the other's choice. If both players choose defection, then they receive the second highest sentence (lowest payoffs), which is less beneficial than what they would receive for mutual cooperation, which constitutes a dilemma.

3b. **Donation game**

modern version of social dilemmas, which reflects the importance of the dilemma

Two players, *x* and *y*, decide independently of each other whether to pay a cost (*c*>0) or not. The returns (*b*>*c*) of the investment go to the coplayer.

$$G = \begin{array}{c} \\ \text{no} \\ \text{yes} \end{array} \begin{array}{c} \begin{array}{cc} \text{no} & \text{yes} \end{array} \\ \begin{pmatrix} (0,0) & (b,-c) \\ (-c,b) & (b-c,b-c) \end{pmatrix} \end{array}$$

The payoffs can be rescaled as:

$$G = \begin{pmatrix} (0,0) & (1+\alpha,-\alpha) \\ (-\alpha,1+\alpha) & (1,1) \end{pmatrix}, \quad \text{where} \quad \alpha = \frac{c}{b-c}$$

3c. **Public goods game**

N players simultaneously decide whether or not to pay €1 into a public pool. The collected sum is multiplied by *r* (<*N*) and shared equally among the players.

For both games, selfish individual interest dictates that players pay nothing [i.e., the NE=(no,no,…)], which constitutes a dilemma.

4. **Rock–paper–scissors game** (cyclic dominance)

After a countdown, the two players each show one of the symbols 'rock', 'paper', or 'scissors' with their hand. Should the players show the same symbol, the game is repeated. Otherwise rock beats scissors, which beat paper, which beats rock.

Payoffs in the symmetric zero-sum version:

$$G = \begin{pmatrix} (0,0) & (1,-1) & (-1,1) \\ (-1,1) & (0,0) & (1,-1) \\ (1,-1) & (-1,1) & (0,0) \end{pmatrix}$$

NE: choose one of the options at random with equal (1/3) probability.

Home assignments

2.1. What is the suggestion of the minimax method for this zero-sum game?

$$G = \begin{pmatrix} (3,-3) & (1,-1) & (-2,2) \\ (2,-2) & (1,-1) & (4,-4) \\ (2,-2) & (-1,1) & (5,-5) \end{pmatrix}.$$

2.2. Find the pure Nash equilibrium of this bimatrix game:

$$G = \begin{pmatrix} (0,0) & (1,2) & (4,5) \\ (5,4) & (1,3) & (0,0) \\ (2,1) & (4,4) & (3,1) \end{pmatrix}.$$

2.3. Determine the mixed Nash equilibria for a) the matching pennies and b) the rock–paper–scissors game defined by:

a) $G = \begin{pmatrix} (1,-1) & (-1,1) \\ (-1,1) & (1,-1) \end{pmatrix}$ and b) $G = \begin{pmatrix} (0,0) & (1,-1) & (-1,1) \\ (-1,1) & (0,0) & (1,-1) \\ (1,-1) & (-1,1) & (0,0) \end{pmatrix}.$

2.4. In the Minority game each of N (odd) players has two options [to buy or to sell, to turn left or right on the way home, to go to the El Farol bar (Santa Fe) or to stay at home, etc.]. The winners will be those who belong to the minority and they receive +1, the others receive -1 payoff. How many pure Nash equilibria does this game have?

2.5. Draw the flow graph of the rock–paper–scissors game!

2.6. Describe 10 real life situations that resemble the prisoner's dilemma, donation, or public goods games!